Chapter 1
Section 1.1

1) $4x+3=0$, $x=-3/4$, rational number system

2) $x^2-x-1=0$, $x=\frac{1\pm\sqrt{5}}{2}$ which is real but irrational. Need real number system

3) $x^2+x+1=0$ $x=\frac{-1\pm\sqrt{-3}}{2}$ need complex number system

4) $\sin x=0$, $x=0$, integers 5) $\cos x=0$, $x=\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots \pm\frac{n\pi}{2}$ real number system n odd

6) $(x+2)(x+1)=0$ $x=-2,-1$, integers

7) $\sin(\log x)=0$, $x=1$, $\sin(\log 1)=0$, integers

8) $z^4=16$, $z=2$, integers 9) $z^4=-16$, $z=(-16)^{1/4}$ complex

10) a) $\left[1+10^{-2}+10^{-4}\dots\right]=\frac{1}{1-10^{-2}}=\frac{10^2}{100-1}=\boxed{\frac{100}{99}}$

b) $23.2323\dots=23\frac{(100)}{99}=\frac{2300}{99}$

c) $376.376376=376\left[1+10^{-3}+10^{-6}+10^{-9}\dots\right]$

$=376\frac{1}{1-10^{-3}}=376\frac{1000}{999}=\frac{376000}{999}$

11 (a) Consider $4.0404\dots=4\left[1+10^{-2}+10^{-4}\dots\right]$

$=4\left[\frac{1}{1-10^{-2}}\right]=\frac{400}{99}$. Now $3.0404\dots$

$=\frac{400}{99}-1=\frac{400-99}{99}=\boxed{\frac{301}{99}}$

b) $.999\dots=.9\left[1+10^{-1}+10^{-2}\dots\right]$

$=.9\frac{1}{1-10^{-1}}=\frac{.9}{1-.1}=\frac{.9}{.9}=1$ q.e.d.

12(a) We begin by showing that the square of any odd number is odd. Let N_0 be that number. Then $N_0=N_e+1$ where N_e is even. Now $N_0^2=\underbrace{N_e^2+2N_e}_{\text{even}}+1$

$N_0^2=\text{even}+1=\text{odd}$

12 (a) <u>continued</u> We showed that the square of an odd integer is odd. Thus the square root of a perfect square (that is even) must not be odd, ∴ is even.

12 (b) $m^2 = 2n^2$, $2n^2$ is even, ∴ from part (a) the square root of $2n^2$, which is m, must be even.

12 (c) $n^2 = \frac{m}{2} * m$. Since m is even, $\frac{m}{2}$ is an integer. ∴ n^2 is even, and from (a) so is n.

12 (d) We assumed that $\sqrt{2} = \frac{m}{n}$ can be expressed as the ratio of 2 integers having no common integer factor. Our assumption says that m and n both can't be even. This resulted in a contradiction, since in parts (b) and (c) we found that m and n were both even

12(e). Assume $n + \sqrt{2}$ is rational $= a$

$a = n + \sqrt{2}$, $a - n = \sqrt{2}$. The left side is rational [the difference of rational numbers] but the right side is irrational. Have a contradiction.

Suppose $\sqrt{2n^2}$ is rational, Then $\sqrt{2}\, n = a$ is rational $\frac{a}{n} = \sqrt{2}$. The left side is the quotient of rational numbers and is rational, the right side is irrational, have a contradiction,

Assume $a = \sqrt{\sqrt{2}}$ is rational $a^2 = \sqrt{2}$.

Left side is rational, right side is irrational. Have contradiction.

13(a) $x^2+bx+c=0$ $\quad x = \frac{-b \pm \sqrt{b^2-4c}}{2} = 1+\sqrt{2}$ $\quad b=-2,\ \frac{\sqrt{4-4c}}{2}=\sqrt{2}$, $c=-1$

$\therefore$ $\boxed{x^2-2x-1=0}$ will work $\quad$ 13(b) $x=\sqrt{\sqrt{2}}$ $\quad x^4=2$ $\quad \boxed{x^4-2=0}$

14 a) Using Matlab: exp(1) = $2.718281828\underbrace{45905}_{\text{pattern}}$ breaks.

Use long format

(b) $201/26 = 7.7\underbrace{307692}_{\text{repeating decimals}}\overbrace{307692}3$ $\quad$ $\boxed{\text{the digits 307692 repeat}}$

15 $\boxed{4-4i}$ $\qquad$ 16 $-5+21+15i+7i = \boxed{16+22i}$

17 $(3-2i)(4+3i)(3+2i) = (3-2i)(3+2i)(4+3i) =$

$(9+4)(4+3i) = \boxed{52+i39}$

chap I

sec 1.1, continued

18| $(1+i)^3 = (1+i)^2(1+i) = 2i\,(1+i) = -2+2i$

Imag part $= \boxed{2}$

19| $\text{Im}(1+i) = 1$, $[\text{Im}(1+i)]^3 = 1^3 = \boxed{1}$

20| $(x+iy)(u-iv)(x-iy)(u+iv) =$

$(x+iy)(x-iy)(u-iv)(u+iv) = (x^2+y^2)(u^2+v^2)$

$= \boxed{u^2x^2+v^2x^2+u^2y^2+v^2y^2}$

21| (a) binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \frac{a^{n-k} b^k \, n!}{(n-k)!\, k!}$$

let $a=1$, $b=iy$

$$(1+iy)^n = \sum_{k=0}^{n} \frac{(iy)^k \, n!}{(n-k)!\, k!}$$

b) $$(1+2i)^5 = \sum_{k=0}^{5} \frac{(i2)^k \, 5!}{(5-k)!\, k!} =$$

$$\frac{5!}{5!} + \frac{(i2)\,5!}{4!\,1!} + \frac{(-4)\,5!}{3!\,2!} + \frac{-i8\,5!}{2!\,3!} + \frac{16\,5!}{1!\,4!} + \frac{i32\cdot 5!}{5!}$$

Real part is $1 + \frac{(-4)(120)}{12} + 80 = 41$

Imag part is $(i2)5 - 8\cdot 10 + 32 = -38$

c) Use $(1+2i)\wedge 5$ in Matlab code.
Will get $41 - i38$

22| $z_1 z_2 = (x_1+iy_1)(x_2+iy_2) = x_1x_2 - y_1y_2 + ix_1y_2 + ix_2y_1$

$\text{Re}(z_1z_2) = x_1x_2 - y_1y_2 = \text{Re}\,z_1\,\text{Re}\,z_2 - \text{Im}\,z_1\,\text{Im}\,z_2$

23| From the above $\text{Im}(z_1z_2) = x_1y_2 + x_2y_1 = \text{Re}\,z_1\,\text{Im}\,z_2 + \text{Re}\,z_2\,\text{Im}\,z_1$

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24| i, $i^2=-1$, $i^3=-i$, $i^4=-i\cdot i=1$

$i^5=i$, etc. $\therefore$ the four possible values are $i, -1, -i, 1$

$i^{n+4}=i^4\, i^n$, but $i^4=1$, $\therefore$ $i^{n+4}=i^n$

25| $i^{1023}=i^{1020}\, i^3 = i^{1020}(-i)$

$= (i^4)^{255}(-i) = 1^{255}(-i) = \boxed{-i}$

26| Find: $(1-i)^{1025}$, Note: $(1-i)^2=-2i$

$(1-i)^{1025} = (1-i)^{1024}(1-i) = \left[(1-i)^2\right]^{512}(1-i)$

$= (-2i)^{512}(1-i) = 2^{512}(-1)^{512}\, i^{512}(1-i)$

$= 2^{512}\left[(i)^4\right]^{128}(1-i) = 2^{512}\, 1^{128}(1-i) = \boxed{2^{512}(1-i)}$

27|

$x=2x$

$\therefore$ $x=0$

$y+1=2y$

$\therefore$ $y=1$, ans $\boxed{x=0,\ y=1}$

28|

$x^2-y^2+i2xy = y+ix$

$x^2-y^2=y$, $2xy=x$ $\Rightarrow$ assume $x\neq 0$, $2y=1$, $y=1/2$

$x^2-\frac{1}{4}=1/2$, $x^2=3/4$, $x=\pm\sqrt{3}/2$

Set of answers $\boxed{x=\pm\sqrt{3}/2,\ y=1/2}$

Now assume $x=0$, $2xy=x$ is satisfied

$x^2-y^2=y$, $-y^2=y$, $y+y^2=0$

$y(1+y)=0$, $y=0$ or $y=-1$

Second set of answer $\boxed{x=0,\ y=0 \text{ or } -1}$

29|

$$e^{x^2+y^2} = e^{-2xy} \qquad 2y = 1, \quad \therefore\ y = \tfrac{1}{2}$$

Now $x^2+y^2 = -2xy$, $\quad x^2+y^2+2xy = 0$

$(x+y)^2 = 0 \quad \therefore\ x = -y, \quad x = -1/2$

answer $\boxed{x = \frac{-1}{2},\ y = 1/2}$

30|

$$\operatorname{Log}(x+y) = 1 \qquad y = xy$$

If $\operatorname{Log}(x+y) = 1$, $\quad x+y = e$

Now assume $y \neq 0 \quad y = xy \quad \Rightarrow x = 1$

Since $x+y = e$, $\quad y = e-1 \quad$ Answer: $\boxed{x=1,\ y=e-1}$

Now assume $y = 0$, $\quad y = xy$ is satisfied

$x+y = e \Rightarrow x = e \qquad$ Answer $\boxed{x=e,\ y=0}$

31|

$$[\operatorname{Log}(x) - 1]^2 = 1, \qquad [\operatorname{Log}(y) - 1]^2 = 0 \quad \therefore \boxed{y=e}$$

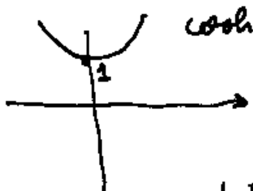
$\operatorname{Log}(x) - 1 = \pm 1$, $\quad \operatorname{Log} x - 1 = -1$

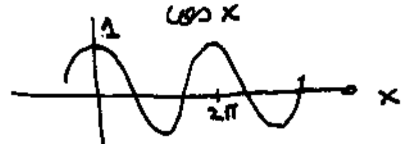
$\operatorname{Log} x = 0, \quad x = 1 \quad$ or $\operatorname{Log} x - 1 = 1$

$\operatorname{Log} x = 2, \quad x = e^2$

Answer $\boxed{y = e,\quad x = 1 \text{ or } e^2}$

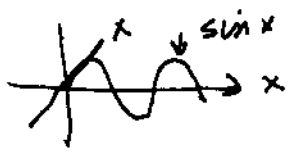
32|





$\cosh(y-1) = \cos x$ can only be solved if $y = 1$ and $x = 2n\pi$, $\quad n = 0, \pm 1, \pm 2 \ldots$

Now need $\sin x = xy \qquad \sin x = x \quad$ (since $y=1$)



$\sin x = x$ if and only if $x = 0$

so Answer $\boxed{x=0,\ y=1}$

Sec 1.2

1) $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$,

$\overline{z_1 - z_2} = (x_1 - x_2) - i(y_1 - y_2) = x_1 - iy_1 - [x_2 - iy_2]$

$= \bar{z}_1 - \bar{z}_2$

2) $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i[y_1 x_2 + y_2 x_1]$

$\overline{z_1 z_2} = (x_1 x_2 - y_1 y_2) - i[y_1 x_2 + y_2 x_1] =$

$(x_1 - iy_1)(x_2 - iy_2) = \bar{z}_1 \bar{z}_2$

3) $\frac{1}{z_1} = \frac{1}{x_1 + iy_1} = \frac{x_1 - iy_1}{x_1^2 + y_1^2}$

$\therefore \overline{\left(\frac{1}{z_1}\right)} = \frac{x_1 + iy_1}{x_1^2 + y_1^2}$ $\quad \frac{1}{\bar{z}_1} = \frac{1}{x_1 - iy_1} = \frac{x_1 + iy_1}{x_1^2 + y_1^2}$

Thus $\overline{\left(\frac{1}{z_1}\right)} = \frac{1}{\bar{z}_1}$

4) $\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{x_2^2 + y_2^2} =$

$\frac{x_1 x_2 + y_1 y_2 + i[y_1 x_2 - y_2 x_1]}{x_2^2 + y_2^2}$. Now $z_1 \frac{1}{z_2}$

$= (x_1 + iy_1)\left[\frac{(x_2 - iy_2)}{x_2^2 + y_2^2}\right] = \frac{x_1 x_2 + y_1 y_2 + i[y_1 x_2 - x_1 y_2]}{x_2^2 + y_2^2}$

Thus $z_1 \frac{1}{z_2} = \left(\frac{z_1}{z_2}\right)$

5) $\overline{\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)} = \overline{\left(\frac{z_1}{z_2}\right)} = \overline{\frac{(x_1 + iy_1)(x_2 - iy_2)}{x_2^2 + y_2^2}}$

$= \overline{\frac{x_1 x_2 + y_1 y_2 + i(y_1 x_2 - y_2 x_1)}{x_2^2 + y_2^2}} = \frac{x_1 x_2 + y_1 y_2 + i[y_2 x_1 - y_1 x_2]}{x_2^2 + y_2^2}$

Now $\frac{\bar{z}_1}{\bar{z}_2} = \frac{x_1 - iy_1}{x_2 - iy_2} = \frac{(x_1 - iy_1)(x_2 + iy_2)}{x_2^2 + y_2^2} = \frac{x_1 x_2 + y_1 y_2 + i[y_2 x_1 - y_1 x_2]}{x_2^2 + y_2^2}$

Thus $\frac{\bar{z}_1}{\bar{z}_2} = \overline{\left(\frac{z_1}{z_2}\right)}$

sec. 1.2 continued

6] $z_1 z_2 = x_1 x_2 - y_1 y_2 + i\,[y_1 x_2 + y_2 x_1]$

$\bar{z}_1\,\bar{z}_2 = x_1 x_2 - y_1 y_2 - i\,[y_1 x_2 + y_2 x_1]$

$\text{Re}\,[z_1 z_2] = \text{Re}\,[\bar{z}_1 \bar{z}_2]$. Alternatively:

$\text{Re}\,[z_1 z_2] = \text{Re}\,[\overline{z_1 z_2}]$ since real part is unaffected by taking conj.

But $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$ from prob 2

$\therefore\ \text{Re}\,[z_1 z_2] = \text{Re}\,[\bar{z}_1 \bar{z}_2]$

7] From part 6] $\quad z_1 z_2 = x_1 x_2 - y_1 y_2 + i\,[y_1 x_2 + y_2 x_1]$

thus: $\quad \bar{z}_1 \bar{z}_2 = x_1 x_2 - y_1 y_2 - i\,[y_1 x_2 + y_2 x_1]$

Note $\text{Im}\,(z_1 z_2) = -\text{Im}\,\bar{z}_1 \bar{z}_2 = y_1 x_2 + y_2 x_1$

8] $\dfrac{1}{1+2i} = \dfrac{1-2i}{1^2+2^2} = \boxed{\dfrac{1}{5} - i\,\dfrac{2}{5}}$

9] $i + \dfrac{1}{1-2i} = i + \dfrac{1+2i}{5} = \dfrac{1}{5} + \dfrac{7i}{5}$

$\left(i + \dfrac{1}{1-2i}\right)^2 = \dfrac{1}{25} + \dfrac{14}{25}\,i - \dfrac{49}{25} =$

$= \boxed{-\dfrac{48}{25} + i\,\dfrac{14}{25}}$

10] $\dfrac{3-4i}{1+2i} = \dfrac{(3-4i)(1-2i)}{5} = \dfrac{1}{5}\left[3 - 8 + i\,[-4-6]\right]$

$= \dfrac{1}{5}\,[-5 - i10] = \boxed{-1-2i}$

11] $\dfrac{3-4i}{1+2i} + \dfrac{3+4i}{1-2i}$. This must equal

$2\,\text{Re}\left[\dfrac{3-4i}{1+2i}\right] = \boxed{-2}$ $\quad$ 12] use ans from 10

12] contd $\quad 2i + \dfrac{3-4i}{1+2i} = 2i + (-1-2i) = \boxed{-1}$

sec 1.2 cont'd

13) $\left(\frac{4-4i}{2+2i}\right)^7 = \left(\frac{2-2i}{1+i}\right)^7 = \left[2^? \left[\frac{(1-i)}{1+i}\right]\right]^7$

$= 2^7 \left[\frac{(1-i)^2}{2}\right]^7 = \frac{2^7}{2^7} \quad (1-i)^{14} =$

$\left[(1-i)^2\right]^7 = (-2i)^7 = (-1)^7\, 2^7\, i^7 = i \cdot 2^7 = \boxed{128i}$

14) $\left(\frac{4-4i}{2+2i}\right)^7 + \left(\frac{4+4i}{2-2i}\right)^7$. Note the second term is the conjugate of the first.

$\therefore$ ans is $2\,\mathrm{Re}\left[\frac{4-4i}{2+2i}\right]^7 = 2\,\mathrm{Re}\left[128i\right]$ (see prob 13)

$= \boxed{0}$

PROBLEM 15

```
» %8
» 1/(1+2i)

ans =

   0.2000 - 0.4000i
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```
» %9
» (i+1/(1-2i))^2

ans =

  -1.9200 + 0.5600i
```

```
» %10
» (3-4i)/(1+2i)

ans =

  -1.0000 - 2.0000i
```

```
» %11
» (3-4i)/(1+2i)+(3+4i)/(1-2i)

ans =

   -2
```

```
» %12
» 2i+(3-4i)/(1+2i)

ans =

    -1
```

```
» %13
» ((4-4i)/(2+2i))^7

ans =

                  0 +1.2800e+002i
```

```
» %14
» ((4-4i)/(2+2i))^7+((4+4i)/(2-2i))^7

ans =

     0
```

16) $\overline{\left(\dfrac{z_1}{z_2 z_3}\right)} = \dfrac{\bar{z}_1}{\overline{z_2 z_3}} = \dfrac{\bar{z}_1}{\bar{z}_2\,\bar{z}_3} = \bar{z}_1 \dfrac{1}{\bar{z}_2 \bar{z}_3}$

this is not equal to $\bar{z}_1 \dfrac{1}{\bar{z}_2 z_3}$ $\therefore$ $\boxed{\text{not true}}$

17) $\overline{z_1 \bar{z}_2 z_3} = \bar{z}_1 \bar{\bar{z}}_2 \bar{z}_3 = \bar{z}_1 z_2 \bar{z}_3$ q.e.d $\boxed{\text{true}}$

18) $\overline{i\,(z_1 + z_2 + z_3)} = \bar{i}\,(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) =$

$-i\,(\bar{z}_1 + \bar{z}_2 + \bar{z}_3)$. This is not equal to $i\,(\bar{z}_1 + \bar{z}_2 + \bar{z}_3)$. $\therefore$ $\boxed{\text{not true}}$

19) Consider $z_1 \bar{z}_2 z_3$. Its conjugate is

$\overline{z_1 \bar{z}_2 z_3} = \bar{z}_1 \bar{\bar{z}}_2 \bar{z}_3 = \bar{z}_1 z_2 \bar{z}_3$

$\therefore \operatorname{Re}[z_1 \bar{z}_2 z_3] = \operatorname{Re}[\bar{z}_1 z_2 \bar{z}_3]$ q.e.d. $\boxed{\text{true}}$

20) Consider $z_1 \bar{z}_2 z_3$. Its conjugate is $\overline{z_1 \bar{z}_2 z_3} = \bar{z}_1 z_2 \bar{z}_3$. If 2 quantities are conjugates of each other, their imag. parts differ in sign. $\therefore \operatorname{Im}(z_1 \bar{z}_2 z_3) = -\operatorname{Im}(\bar{z}_1 z_2 \bar{z}_3)$

Note the minus sign. $\therefore$ the result given in the problem is $\boxed{\text{not true}}$.

21| Observe that $\bar{z}_1 z_2 \bar{z}_3$ is the conjugate of $z_1 \bar{z}_2 z_3$. Suppose $z = a + ib$

$i\bar{z} = i(a - ib) = b + ia$. Note $\mathrm{Re}(z) = \mathrm{Im}(i\bar{z})$

$\therefore \mathrm{Re}(z_1 \bar{z}_2 z_3) = \mathrm{Im}(i \bar{z}_1 z_2 \bar{z}_3)$. Is $\boxed{\text{true}}$

22|

a) If $p + iq = (k + il)(m + in)$

$\overline{p+iq} = \overline{(k+il)}\ \overline{(m+in)}$

$p - iq = (k - il)(m - in)$

Now if $(p + iq) = (k + il)(m + in) = km - ln + i(lm + kn)$

Equate reals

$p = km - ln$, require $p \geq 0$ $\therefore$ take

$p = |km - ln|$ since only p^2 is of interest.

Now equate imaginaries in the above:

$q = lm + kn$ which means

that $(p + iq) = (k + il)(m + in)$ is satisfied and so is $(p - iq) = (k - il)(m - in)$ and so is $(p^2 + q^2) = (k^2 + l^2)(m^2 + n^2)$

b) $(p^2 + q^2) = (p + iq)(p - iq) = (k + il)(m - in)(k - il)(m + in)$

Take $(p + iq) = (k + il)(m - in)$. $(p - iq) = (k - il)(m + in)$

The second eqn. is the conjugate of the first. So if the 1st is satisfied, so is the second.

$p + iq = km + ln + i[lm - kn]$. Take $p = km + ln$

$q = |lm - kn|$ [sign is made pos.].

prob 22] cont'd

c] $(3^2+5^2)(2^2+7^2) = p^2+q^2$

$k=3,\ l=5,\ m=2,\ n=7$

$p = |km-nl| = 29,\ q = lm+kn = 31$

$29^2+31^2 = (3^2+5^2)(2^2+7^2) = 1802$

Try $p = km+nl = 41,\ q = |lm-kn| = 11$

$p^2+q^2 = (122)(53) = (11^2+1^2)(7^2+2^2) = 6466$

Take $k=11,\ l=1,\ m=7,\ n=2$

$\therefore\ |km-nl| = 75,\ lm+kn = 29,$ $\boxed{p=75,\ q=29}$

Note $75^2+29^2 = 6466$

$km+nl = 79;\ |lm-kn| = 15$ $\boxed{p=79,\ q=15}$

23] a) $\dfrac{(c,d)}{(a,b)} = (e,f)$ $\qquad (c,d) = (a,b)(e,f)$

$(c,d) = (ae-bf,\ be+af)$. (b) Thus

$ae-bf = c$
$be+af = d$

(c) Apply Cramer's rule to this pair of linear simult. equations with unknowns e, f. assume $a\neq 0,\ b\neq 0$

$$e = \begin{bmatrix} c & -b \\ d & a \end{bmatrix} \Big/ (a^2+b^2) = \boxed{\frac{ac+bd}{a^2+b^2}} = e \qquad a^2+b^2\neq 0$$

$$f = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \Big/ (a^2+b^2) = \boxed{\frac{ad-bc}{a^2+b^2}} = f$$

$$\frac{c+id}{a+ib} = \frac{(c+id)(a-ib)}{a^2+b^2} = \frac{ac+bd+i(ad-bc)}{a^2+b^2}$$

$= e+if$. Thus. $e = \dfrac{ac+bd}{a^2+b^2},\quad f = \dfrac{ad-bc}{a^2+b^2}$

(same result). [assuming $a^2+b^2\neq 0$]

Prob 24 For $\frac{p}{q} = \frac{r}{s}$ require $q \neq 0$ and $s \neq 0$

mult both sides by qs

$$\frac{p\,qs}{q} = \frac{r}{s}\,qs \qquad ps = qr$$

Summary: $\frac{p}{q} = \frac{r}{s}$ Necessary and sufficient

conditions : $\boxed{q \neq 0 \text{ and } s \neq 0 \quad \text{and} \quad ps = qr}$

1] $|3-i| = \sqrt{3^2+1^2} = \boxed{\sqrt{10}}$

2] $|(2i)(3+i)| = |2i|\,|3+i| = 2\sqrt{3^2+1} = \boxed{2\sqrt{10}}$

3] $|(2-3i)(3+i)| = |2-3i|\,|3+i| = \sqrt{2^2+3^2}\sqrt{3^2+1^2}$
$= \sqrt{13}\,\sqrt{10}$
$= \boxed{\sqrt{130}}$

4] $|(2-3i)^2(3+3i)^3| = |(2-3i)^2|\,|(3+3i)^3|$

$= (4+9)\left(\sqrt{9+9}\right)^3 = \boxed{13\cdot\left(\sqrt{18}\right)^3} \doteq \boxed{993}$

5] $|2i+2i(3+i)| = |2i+6i-2| = |-2+8i|$

$= \sqrt{4+64} = \sqrt{68} = \boxed{2\sqrt{17}}$

6] $\left|1+i+\frac{1}{1+i}\right| = \left|(1+i)+\frac{(1-i)}{2}\right| = \left|\frac{3}{2}+\frac{i}{2}\right| = \sqrt{\frac{9}{4}+\frac{1}{4}}$

$= \boxed{\frac{\sqrt{10}}{2}}$

7] $\left|\frac{(1+i)^5}{(2+3i)^5}\right| = \left|\frac{1+i}{2+3i}\right|^5 = \left|\frac{\sqrt{2}}{\sqrt{13}}\right|^5 = \boxed{\left[\sqrt{\frac{2}{13}}\right]^5 \doteq .0093}$

8] $\left|\frac{(1-i)^n}{(2+2i)^n}\right| = \left|\frac{(\sqrt{2})^n}{2^n\,|1+i|^n}\right| = \left|\frac{(\sqrt{2})^n}{2^n(\sqrt{2})^n}\right| = \boxed{\frac{1}{2^n}}$

9] $\left|\frac{1}{(1-i)}+\frac{1}{(1+i)}+\frac{5}{(1+2i)}\right| = \left|\frac{1+i}{2}+\frac{1-i}{2}+\frac{5(1-2i)}{5}\right|$

$= |1+1-2i| = |2-2i| = \boxed{2\sqrt{2}}$

10| $|\alpha + i\beta + \alpha - i\beta| = 1$

$|2\alpha| = 1$ take $\alpha = 1/2$

$\sqrt{\alpha^2+\beta^2} + \sqrt{\alpha^2+\beta^2} = 2$ $\sqrt{\alpha^2+\beta^2} = 1$

$\sqrt{\frac{1}{4}+\beta^2} = 1$ $1/4 + \beta^2 = 1$, $\beta^2 = 3/4$

$\beta = \pm\sqrt{3}/2$. Take ans. as $\boxed{\frac{1}{2}+i\frac{\sqrt{3}}{2}}$

and $\boxed{\frac{1}{2} - i\frac{\sqrt{3}}{2}}$

11| $z_1 = \alpha + i\beta$, $z_2 = \alpha - i\beta$

$|z_1 + z_2| = a$ $2|\alpha| = a$. Try $\alpha = a/2$

$|z_1| + |z_2| = \frac{1}{a}$ $\sqrt{\alpha^2+\beta^2} + \sqrt{\alpha^2+\beta^2} = \frac{1}{a}$

$2\sqrt{\alpha^2+\beta^2} = \frac{1}{a}$ $\alpha^2+\beta^2 = \frac{1}{4a^2}$

$\frac{a^2}{4} + \beta^2 = \frac{1}{4a^2}$ $\beta^2 = \frac{1}{4}\left[\frac{1}{a^2} - a^2\right]$ > 0

$\beta = \pm\frac{1}{2}\sqrt{\frac{1}{a^2} - a^2}$

answer: $z_1 = \frac{a}{2} + \frac{i}{2}\sqrt{\frac{1}{a^2} - a^2}$

$z_2 = \frac{a}{2} - \frac{i}{2}\sqrt{\frac{1}{a^2} - a^2}$

Check $|z_1 + z_2| = a$

$\sqrt{\frac{a^2}{4} + \frac{1}{4a^2} - \frac{a^2}{4}} + \sqrt{\frac{a^2}{4} + \frac{1}{4}\left(\frac{1}{a^2} - a^2\right)}$

$= 2\sqrt{\frac{1}{4a^2}} = \frac{1}{a}$

12) Because their difference is purely imaginary, their real parts must be identical. Let the numbers be

$z_1 = x + iy$ and $z_2 = x + i\beta$

Now $z_1 - z_2 = i$ $\therefore$ $y - \beta = 1$

$z_1 z_2 = x^2 - y\beta + ix[y + \beta] = 2$

Now $x[y + \beta] = 0$

either $x = 0$ or $y = -\beta$

suppose $x = 0$, then using $z_1 z_2 = 2$

have $-y\beta = 2$, $\beta = -2/y$

Since $y - \beta = 1$ have $y + \frac{2}{y} = 1$

Use quadratic formula, has no real sol'n.

So Take $y = -\beta$, Now $y - \beta = 1$

$y + y = 1$, $y = 1/2$, $\beta = -1/2$

Recall $x^2 - y\beta = 2$ $\quad x^2 + \frac{1}{4} = 2$

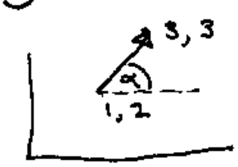
$x = \pm\sqrt{7}/2$

answers : $\boxed{\sqrt{7}/2 + i/2, \quad \sqrt{7}/2 - i/2}$

also $\boxed{-\sqrt{7}/2 + i/2, \quad -\sqrt{7}/2 - i/2}$

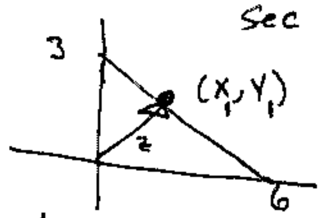
13) $1 - (-1) + i[4 - (-3)] = \boxed{2 + i7}$

14) $5\cos 30^\circ + i\,5\sin 30^\circ = \boxed{5\frac{\sqrt{3}}{2} + i\frac{5}{2}}$

15) 

$\cos\alpha = \frac{2}{\sqrt{5}}$, $\sin\alpha = \frac{1}{\sqrt{5}}$

$a = 5\cos\alpha = \boxed{2\sqrt{5} = a}$ $\quad 5\sin\alpha = \boxed{\sqrt{5} = b}$

16] 

$x + 2y = 6$

$z = a + ib$ or

$z = x_1 + iy_1$ $\quad x_1 + 2y_1 = 6, \quad y_1 = 3 - \frac{x_1}{2}$

$z = x_1 + i\left[3 - \left(\frac{x_1}{2}\right)\right]$

slope of given line is $-1/2$

slope of vector is $\frac{3 - x_1/2}{x_1}$ must $= 2$

two slopes are neg. recips

$3 - x_1/2 = 2x_1$; $\boxed{x_1 = \frac{6}{5}}$ $\quad y_1 = 3 - \frac{x_1}{2} = 3 - \frac{3}{5} = 2\,2/5$

answer: $\boxed{\frac{6}{5} + i\,\frac{12}{5}}$ $\quad \boxed{y_1 = \frac{12}{5}}$

17] Let $x_1 + iy_1$ be the vector $= a + ib$

$(x_1 - 1)^2 + y_1^2 = 1$

$x_1^2 - 2x_1 + 1 + y_1^2 = 1$

Now $\sqrt{x_1^2 + y_1^2} = 3/2$

use here $\quad x_1^2 + y_1^2 = 9/4$

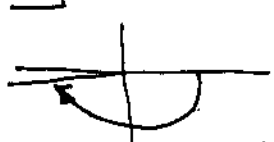
$\therefore \quad -2x_1 + 1 + 9/4 = 1$ $\qquad \boxed{9/8 = x_1}$

Now $\left(\frac{9}{8}\right)^2 + y_1^2 = 9/4$

$\sqrt{9/4 - (9/8)^2} = y_1$ $\qquad \boxed{y_1 = \frac{3}{8}\sqrt{7}}$

18] ans is $\boxed{3.14} = \theta$ since $-\pi < \theta \le \pi$

19] Notice the angle 3.15 exceeds π



subtract off 2π

$3.15 - 2\pi = \boxed{-3.133}$

Note $-\pi < \theta \le \pi$

20] $-3 \text{ cis}(3.14) = 3 \text{ cis } (3.14+\pi)$
The angle $3.14+\pi$ does not satisfy
$-\pi < \theta \le \pi$, But we can subtract off 2π
Use $3.14+\pi-2\pi = 3.14-\pi = \boxed{-.001593}$
The preceding is the princ. value.

21] $-4 \text{ cis}(73.7\pi) = 4 \text{ cis}(74.7\pi)$
can subtract 74π from argument
get $\boxed{\theta = .7\pi}$ = princ. value.

22] $3 \text{ cis}(1.1\pi) * 4 \text{ cis}(1.2\pi) = 12 \text{ cis } (2.3\pi)$
subtract 2π from arg. get $\boxed{.3\pi}$

23] $\dfrac{3 \angle 1.5\pi}{3 \angle -1.57} = 1 \angle 3.14$
$-\pi < 3.14 \le \pi$, 3.14 is a princ. value

24] $\dfrac{3 \angle 1.57}{3 \angle -1.58} = 1 \angle 3.15$ 3.15 not a
princ. value, but we can subtract 2π
$3.15 - 2\pi = \boxed{-3.133}$

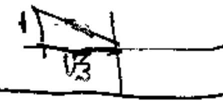
25] $5 \text{ cis } (-98.5\pi)$ we can add 98π
to the argument get $-.5\pi = \boxed{-\pi/2}$

26] $5 \text{ cis } (3\pi^2) = 5 \text{ cis } 29.61$ does not
lie between $-\pi$ and π. Subtract 10π
$29.61 - 10\pi = \boxed{-1.81}$

27] $3[\cos 4 + i \sin(-4)] = \boxed{-1.96 + i\,2.27}$

28] $4\left[\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right] = 2\sqrt{2} + i\,2\sqrt{2}$

29] $r = \sqrt{3+1} = 2$



$\theta = \frac{5\pi}{6}$ princ.

$\boxed{2 \angle \frac{5\pi}{6} + 2k\pi}$

$\boxed{\text{p.v. when } k = 0}$

30] $(1+i)(-\sqrt{3}+i) = \sqrt{2} \angle \frac{\pi}{4} \; 2 \angle \frac{5\pi}{6}$

$= 2\sqrt{2} \angle \frac{13\pi}{12} = 2\sqrt{2} \angle \frac{-11\pi}{12}$

$\boxed{2\sqrt{2} \angle \frac{-11\pi}{12} + 2k\pi}$ p.v. when $k=0$

31] $\sqrt{2} \angle \frac{-3\pi}{4} \left(2 \angle +\frac{5}{6}\pi\right)^3$

$= \sqrt{2} * 8 \angle -\frac{3\pi}{4} + \frac{15\pi}{6} =$

$8\sqrt{2} \angle \frac{-3\pi}{4} + 2\frac{1}{2}\pi = 8\sqrt{2} \angle \frac{-3\pi}{4} + \frac{\pi}{2}$

$= 8\sqrt{2} \angle \frac{-\pi}{4} = \boxed{8\sqrt{2} \angle \frac{-\pi}{4} + 2k\pi}$ $\boxed{\text{p.v. } k=0}$

32] $(-4+3i)^2 = 7 - 24i = \sqrt{7^2+24^2} \angle -\tan^{-1}\frac{24}{7} + 2k\pi$

$= \boxed{25 \angle -1.287 + 2k\pi}$ p.v. when $k=0$.

33] angle $(z1 * z2) = \boxed{2.879}$

angle (z_1) + angle $(z_2) = \boxed{2.879}$

angle $(z1 * z3) = -2.8797$, angle z_1 + angle $z_3 = \boxed{3.40}$

angle $(z1 * z3) \neq$ angle z_1 + angle z_3 because

angle z_1 + angle z_3 $= 3.4$ not a princ value, but $\boxed{-2.87}$ is princ. value

34]

$$\frac{-1-i}{\sqrt{3}+i} = \frac{\sqrt{2}\angle\frac{-3\pi}{4}}{2\angle\pi/6} = \frac{1}{\sqrt{2}}\angle -\frac{22\pi}{24}$$

$$= \boxed{\frac{1}{\sqrt{2}}\angle\frac{-11\pi}{12}}$$

35] $$\frac{\sqrt{2}\angle\frac{-3\pi}{4}\ \ 1\angle\pi/4}{(2\angle 30^\circ)^2} = \frac{\sqrt{2}\angle\frac{-\pi}{2}}{4\angle\pi/3} = \boxed{\frac{1}{2\sqrt{2}}\angle\frac{-5\pi}{6}}$$

36] $$\frac{\text{cis}\,(2\pi)}{\text{cis}\,(-4\pi/3)} = \text{cis}\left[\frac{6\pi}{3}+\frac{4\pi}{3}\right] = \text{cis}\left[\frac{10\pi}{3}\right]$$

$$= \text{cis}\left[3\tfrac{1}{3}\pi\right] = \text{cis}\left[-\frac{2}{3}\pi\right]$$

37]

Eqn (1.3-7) $\quad |z_1+z_2| \leq |z_1| + |z_2|$.

If vectors z_1 and z_2 are both pointing in same direction, then equality will hold. (They cannot point in opposite directions). Thus for equality $\arg z_1 = \arg z_2 + 2k\pi$ where k is any integer.

38]

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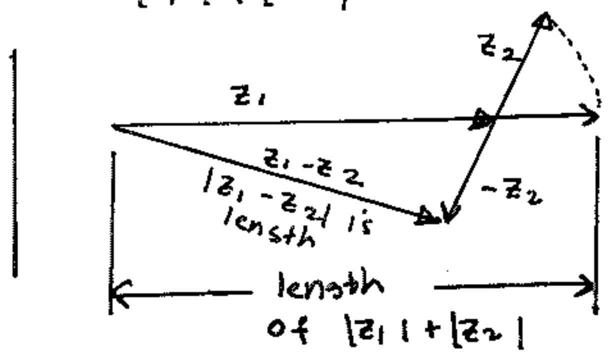
38 a) Given: $|z_1 + z_2| \le |z_1| + |z_2|$

Use $-z_2$ in place of z_2

$|z_1 - z_2| \le |z_1| + |-z_2|$ but $|-z_2| = |z_2|$

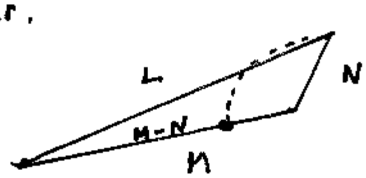
$\therefore$ $|z_1 - z_2| \le |z_1| + |z_2|$



The leg of the triangle with length $|z_1 - z_2|$ must be $\le$ sum of lengths of remaining two legs, i.e. $|z_1| + |z_2|$

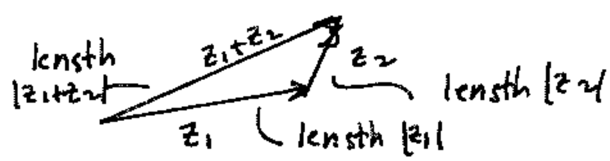
(b) If z_1 and $-z_2$ are in same direction equality will hold, i.e. z_1 and z_2 are in opposite directions, i.e. $\arg z_1 = \arg z_2 + \pi + 2k\pi$, k is an integer.

39 a)



From figure, $L \ge M - N$ $\boxed{M \ge N}$

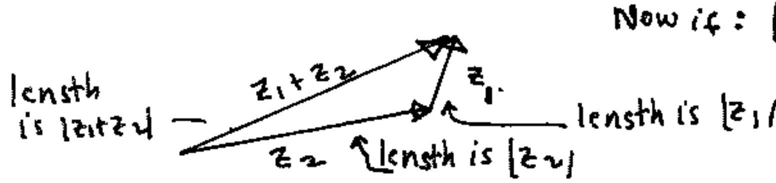
(b)



If: $|z_1| \ge |z_2|$

Referring to part (a) have $|z_1 + z_2| \ge |z_1| - |z_2|$

Now if: $|z_2| \ge |z_1|$



$|z_1 + z_2| \ge |z_2| - |z_1|$,

[continued next pg]

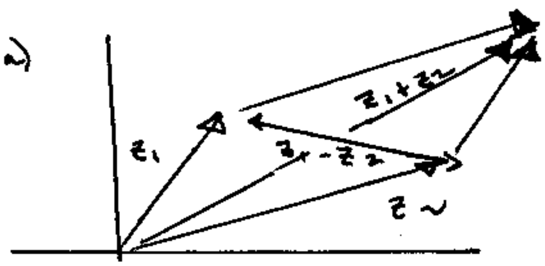
Explanation Eqn (1.3-20)

If $|z_2| \geq |z_1|$ then $\big||z_2|-|z_1|\big| = |z_2|-|z_1|$

If $|z_1| \geq |z_2|$ then $\big||z_2|-|z_1|\big| = |z_1|-|z_2|$

Thus by using $\big||z_2|-|z_1|\big|$ we always get the required right hand side.

40| a)



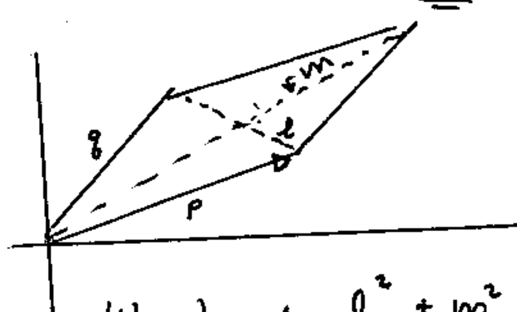
$|z_1-z_2|^2 + |z_1+z_2|^2 =$

$(\bar{z}_1-\bar{z}_2)(z_1-z_2) + (\bar{z}_1+\bar{z}_2)(z_1+z_2) =$

$|z_1|^2+|z_2|^2 - \bar{z}_1 z_2 - \bar{z}_2 z_1 + |z_1|^2+|z_2|^2 + \bar{z}_1 z_2 + z_1\bar{z}_2$

$= 2|z_1|^2 + 2|z_2|^2$ q.e.d

b)



$l = |z_1 - z_2|$

$m = |z_1+z_2|$

$p = |z_2|,\ q = |z_1|$

We showed $l^2 + m^2 = 2(p^2+q^2)$

41|(a) $(p-q)^2 \geq 0$ since $p-q$ is real.

$p^2 - 2pq + q^2 \geq 0$

$p^2+q^2 \geq 2pq$ reverse this

$2pq \leq p^2+q^2$, add p^2+q^2 to both sides

$p^2+q^2+2pq \leq 2p^2+2q^2$

$(p+q)^2 \leq 2(p^2+q^2)$ take square root both sides.

$p+q \leq \sqrt{2}\sqrt{p^2+q^2}$ q.e.d.

(b) Take $p = |\text{Re } z|$, $q = |\text{Im } z|$, $p^2 + q^2 = |\text{Re } z|^2 + |\text{Im } z|^2$
$= |z|^2$. Thus using the formula of part (a)
we have: $|\text{Re } z| + |\text{Im } z| \leq \sqrt{2}\sqrt{|z|^2}$
or $|\text{Re } z| + |\text{Im } z| \leq \sqrt{2}\,|z|$.

(c) $1 + \sqrt{3} \leq \sqrt{2}\ \sqrt{1^2 + (\sqrt{3})^2}$. The left side of the preceding is $\approx 2.732\ldots$. The right side is $\sqrt{2} * 2 \approx 2.828$. Since $2.732.. < 2.828$ the inequality has worked.

(d). For equality $(p-q)^2 = 0$, $\therefore p = q$. $\therefore |\text{Re } z| = |\text{Im } z|$
Take $z = 1+i$, then $|\text{Re } z| + |\text{Im } z| = 2$ while $\sqrt{2}\,|z|$
$= \sqrt{2}\ \sqrt{1^2+1^2} = 2$, and $2 = 2$.

42) $\dfrac{1+i}{\sqrt{3}+i} = \dfrac{(1+i)(\sqrt{3}-i)}{4} = \dfrac{1}{4}\left[\sqrt{3}+1 + i\left[\sqrt{3}-1\right]\right]$

Note, the argument of this is $\tan^{-1} \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$
which must equal $\dfrac{\pi}{12}$. Q.e.d

43a) $(1+ia)(1+ib) = 1 - ab + i(a+b)$

$\arg[(1+ia)(1+ib)] = \arg(1+ia) + \arg(1+ib)$ [1]

$\arg[(1+ia)(1+ib)] = \arg[1 - ab + i(a+b)]$ [2]

$\arg(1+ia) = \tan^{-1} a$, $\arg(1+ib) = \tan^{-1}(b)$ [3]

$\arg[1 - ab + i(a+b)] = \tan^{-1} \dfrac{a+b}{1-ab}$ [4]

Use [4] and [3] in [2] and [1]

Get $\tan^{-1} a + \tan^{-1} b = \tan^{-1}\left(\dfrac{a+b}{1-ab}\right)$

b) set $a = 1/2$, $b = 1/3$

$$\tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{3} = \tan^{-1}\left[\frac{1/2 + 1/3}{1 - 1/6}\right] = \tan^{-1}(1)$$

$\tan^{-1}(1) = \pi/4$. Thus $\pi = 4 \cdot \left[\tan^{-1}\left(\tfrac{1}{3}\right) + \tan^{-1}\left(\tfrac{1}{2}\right)\right]$

c) $(1+ia)(1+ib)(1+ic) = (1 - ab + i(a+b))(1+ic) =$
$1 - ab - ac - bc + i[a+b+c-abc]$

$$\arg[(1+ia)(1+ib)(1+ic)] = \underbrace{\arg[1-ab-ac-bc+i[a+b+c-abc]]}_{\text{this.}}$$

$\arg(1+ia) + \arg(1+ib) + \arg(1+ic) =$

$$\tan^{-1}a + \tan^{-1}b + \tan^{-1}c = \tan^{-1}\left[\frac{a+b+c-abc}{1-ab-ac-bc}\right]$$

44

a) $|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)} = (z_1+z_2)(\bar{z}_1+\bar{z}_2)$

$$= z_1\bar{z}_1 + z_2\bar{z}_2 + \overbrace{z_1\bar{z}_2 + z_2\bar{z}_1}^{2\mathrm{Re}(z_1\bar{z}_2)} = |z_1|^2 + |z_2|^2 + 2\mathrm{Re}[z_1\bar{z}_2]$$

Thus $|z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2\mathrm{Re}[z_1\bar{z}_2]$ (1)

Now: $|\mathrm{Re}\, z_1\bar{z}_2| \leq |z_1\bar{z}_2| = |z_1||z_2|$. This implies:

$|z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1\bar{z}_2) \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$ (2)

Combining (2) and (1) we have:

$$|z_1+z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \quad \text{q.e.d}$$

b) From the preceding $|z_1+z_2|^2 \leq (|z_1| + |z_2|)^2$

Now take pos. square root both sides:

$|z_1+z_2| \leq |z_1| + |z_2|$

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45] $\boxed{\text{Sec. 1.3 continued}}$

a) $(z_1 - z_2)\overline{(z_1 - z_2)} = |z_1 - z_2|^2 =$

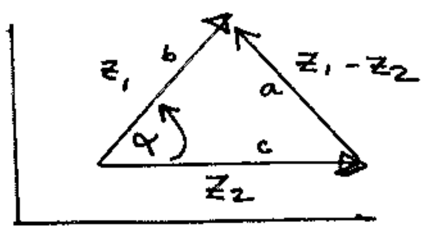
$(z_1 - z_2)(\bar{z}_1 - \bar{z}_2) = |z_1|^2 + |z_2|^2 - z_2\bar{z}_1 - z_1\bar{z}_2$

$= |z_1|^2 + |z_2|^2 - [z_1\bar{z}_2 + \bar{z}_1 z_2]$. Note:

$z_1\bar{z}_2 + \bar{z}_1 z_2 = z_1\bar{z}_2 + \overline{z_1\bar{z}_2} = 2\operatorname{Re}(z_1\bar{z}_2)$

Thus have $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\operatorname{Re}(z_1\bar{z}_2)$. q.e.d.

b)



z_2 is pos. real

$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\operatorname{Re}(z_1\bar{z}_2)$

Take $|z_1 - z_2| = a$, $|z_1| = b$, $z_2 = |z_2| = c$, $\bar{z}_2 = c$

Thus have $a^2 = b^2 + c^2 - 2\operatorname{Re}(z_1 c)$. Now

$\operatorname{Re}(z_1 c) = c\operatorname{Re} z_1 = c|z_1|\cos\alpha = cb\cos\alpha$.

Thus $a^2 = b^2 + c^2 - 2bc\cos\alpha$ q.e.d

sec 1.4

1]  $-\sqrt{3} - i = 2 \angle -\frac{5\pi}{6}$

$\left[2 \angle -\frac{5\pi}{6}\right]^7 = 2^7 \angle \frac{-35\pi}{6} = 2^7 \angle -5\frac{5}{6}\pi$

add 6π to angle $= 2^7 \angle \frac{\pi}{6} = \boxed{128 \angle \pi/6}$

2] $(1+i)^3 (\sqrt{3}+i)^3 = \left[\sqrt{2} \angle \pi/4\right]^3 \left[2 \angle \pi/6\right]^3$

$= 2\sqrt{2} \cdot 8 \angle \frac{3\pi}{4} \angle \pi/2 = 16\sqrt{2} \angle \frac{5\pi}{4}$

$= \boxed{\sqrt{2} \cdot 16 \angle -3\pi/4}$

3] Let $\theta = -\tan^{-1}\frac{4}{3}$, $(3-4i)^6 = 5^6 \angle 6\theta$

$6\theta = -5.5638$ $\quad (3-4i)^6 = 5^6 \angle -5.5638$

-5.5638 not prin. val., add on 2π, get $.7194$

ans $\boxed{5^6 \angle .7194}$

4] $(1 - i\sqrt{3})^{-7} = (2 \angle -\pi/3)^{-7} = 2^{-7} \angle \frac{7\pi}{3}$

$7\pi/3 = 2\frac{1}{3}\pi$, not princ. value, subtr. 2π

get $\frac{1}{2^7} \angle \pi/3 = \boxed{\frac{1}{128} \angle \pi/3}$

5] $(3+4i) = 5 \angle \tan^{-1}\frac{4}{3}$, $(3+4i)^{-6} = 5^{-6} \angle -6 * \tan^{-1}\frac{4}{3}$

$= 5^{-6} \angle -5.5638$, angle is not princ

add 2π, ans $\boxed{\frac{1}{5^6} \angle .7194}$

6] $(\cos\theta + i\sin\theta)^3 \overset{\boxed{\text{sec } 1.4}}{=} \cos 3\theta + i \sin 3\theta$

$= \cos^3\theta + 3\cos^2\theta (i)(\sin\theta) + (3)(\cos\theta)(-\sin^2\theta) - i\sin^3\theta$

$= \cos 3\theta + i \sin 3\theta$

a) Equate imaginaries $3\cos^2\theta\sin\theta - \sin^3\theta = \sin 3\theta$

b) Equate real: $\boxed{\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta}$

7] a) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

use binomial theorem

$$\sum_{k=0}^{n} (\cos\theta)^{n-k}(\sin\theta)^k i^k \frac{n!}{(n-k)!\,k!} = \cos n\theta + i\sin n\theta$$

Equate reals:

$$\cos n\theta = \text{Real} \sum_{k=0}^{n} (\cos\theta)^{n-k} i^k (\sin\theta)^k \frac{n!}{(n-k)!\,k!}$$

$\sin\theta = \sqrt{1-\cos^2\theta}$

$$\rightarrow \cos n\theta = \text{Real} \sum_{k=0}^{n} (\cos\theta)^{n-k}\left(\sqrt{1-\cos^2\theta}\right)^k i^k \frac{n!}{(n-k)!\,k!}$$

b) Now if k is odd i^k is imaginary

if k is even, i^k is real

Thus sum only even values of k. Suppose n is even, we let $k = 2m$, $m = 1, 2, \cdots \frac{n}{2}$.

$i^k = (-1)^m$ since $k = 2m$ and $i^{2m} = (i^2)^m = (-1)^m$.

$$\cos n\theta = \sum_{m=0}^{n/2} \cos^{n-2m}\left[\sqrt{1-\cos^2\theta}\right]^{2m} (-1)^m \frac{n!}{(n-2m)!\,(2m)!}$$

Note $\left(\sqrt{1-\cos^2\theta}\right)^{2m} = (1-\cos^2\theta)^m$

If n is odd, we need only sum as far as $\frac{n-1}{2}$ on m

$$\text{Thus } \cos n\theta = \sum_{m=0}^{\left(\frac{n-1}{2}\right)} \cos^{n-2m}\left[1-\cos^2\theta\right]^m (-1)^m \frac{n!}{(n-2m)!\,(2m)!}$$

c) Take $n=4$

$$\cos 4\theta = \sum_{m=0}^{2} \cos^{4-2m}\theta \left[1-\cos^2\theta\right]^m (-1)^m \frac{4!}{(4-2m)!\,(2m)!}$$

$= \cos^4\theta + \cos^2\theta\left[1-\cos^2\theta\right](-1)\frac{4!}{(2!)^2} + (1-\cos^2\theta)^2 = 8\cos^4\theta - 8\cos^2\theta + 1$

Sec 1.4

7(d) $T_5(x) = \sum_{m=0}^{2} (x^{5-2m})(1-x^2)^m (-1)^5 \frac{5!}{(5-2m)!\,(2m)!}$

$= x^5 + x^3(1-x^2)(-1)\frac{5!}{3!\,2!} + x(1-x^2)^2\frac{5!}{4!}$

$= 16x^5 - 20x^3 + 5x$

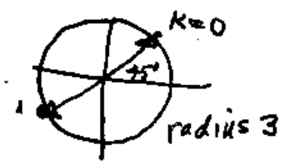
8) $\frac{(\cos\theta + i\sin\theta)}{(\cos\theta - i\sin\theta)^n} = \frac{\cos n\theta + i\sin n\theta}{[\cos -\theta + i\sin(-\theta)]^n}$

$= \frac{\cos n\theta + i\sin n\theta}{\cos(-n\theta) + i\sin(-n\theta)} = \frac{\cos n\theta + i\sin n\theta}{\cos n\theta - i\sin n\theta}$

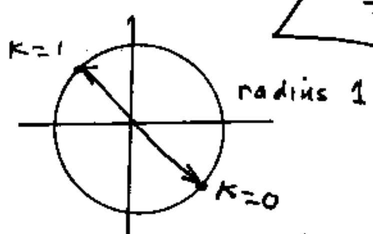
$= \frac{\cancel{\cos n\theta}\,[1 + i\tan n\theta]}{\cancel{\cos n\theta}\,[1 - i\tan n\theta]}$ $\boxed{q.e.d}$

9) $(9i)^{1/2} = \sqrt{9}\ \angle \frac{\pi/2}{2} + \frac{2k\pi}{2}$ $k=0,1$

$= 3\ \angle \frac{\pi}{4} + k\pi = \boxed{\pm\left(\frac{3}{\sqrt{2}} + i\frac{3}{\sqrt{2}}\right)}$



10) $i^{-1/2} = 1\ \angle \frac{\pi}{2}\left(\frac{-1}{2}\right) + \frac{k\pi 2}{-2}$ $k=0,1$



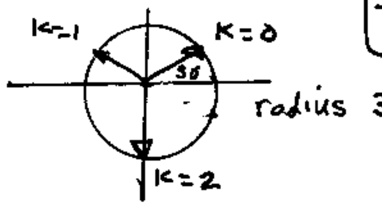
$= \boxed{\pm\left[\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]}$

11) $(27i)^{1/3} = \sqrt[3]{27}\ \angle \frac{\pi}{2}\frac{1}{3} + \frac{2k\pi}{3}$ $k=0,1,2$

$= 3\ \angle \frac{\pi}{6} + \frac{2k\pi}{3}$ $k=0,1,2$

$= 3\cos\frac{\pi}{6} + i3\sin\frac{\pi}{6} =$

$2.59 + i\,1.5$	$k=0$
$-2.59 + i\,1.5$	$k=1$
$-3i$	$k=2$



sec 1.4 cont'd

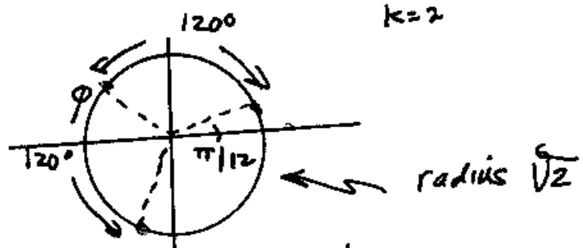
12

$$(1+i)^{1/3} = \sqrt[3]{\sqrt{2}}\ \angle \frac{\pi}{12} + \frac{2k\pi}{3} \qquad k=0,1,2$$

$$= \sqrt[6]{2}\ \left[\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right]_{k=0} = \boxed{1.0842 + i\,.2905}$$

$$= \sqrt[6]{2}\ \left[\cos\left[\frac{3\pi}{4}\right] + i\sin\left[\frac{3\pi}{4}\right]\right]_{k=1} = \boxed{-.7937 + i\,.7937}$$

$$= \sqrt[6]{2}\ \left[\cos\left[\frac{17\pi}{12}\right] + i\sin\left[\frac{17\pi}{12}\right]\right]_{k=2} = -.2905 - i\,1.0842$$



13

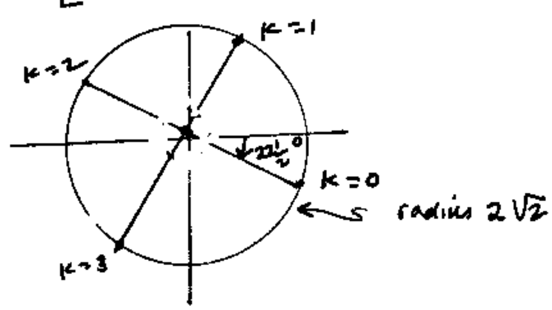
$$(-64i)^{1/4} = \sqrt[4]{64}\ \angle \frac{-\pi}{8} + \frac{2k\pi}{4} \qquad k=0,1,2,3$$

$$= 2\sqrt{2}\left[\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right] \ [k=0] = \boxed{2.6131 - i\,1.0824}$$

$$= 2\sqrt{2}\left[\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right] \ (k=1) = 1.0824 + i\,2.6131$$

$$= 2\sqrt{2}\left[\cos\frac{7\pi}{8} + i\sin\frac{7\pi}{8}\right] \ (k=2) = -2.6131 + i\,1.0824$$

$$= 2\sqrt{2}\left[\cos\frac{11\pi}{8} + i\sin\frac{11\pi}{8}\right] \ (k=3) = \boxed{-1.0824 - i\,2.6131}$$



14] $-\sqrt{3}+i = 2\angle\frac{5\pi}{6}$

$$(-\sqrt{3}+i)^{-1/5} = \frac{1}{\sqrt[5]{2}} \angle \frac{5\pi}{6} + \frac{2k\pi}{-5} \qquad k=0,1,2,3,4$$

$$= \frac{1}{\sqrt[5]{2}}\left[\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right] \quad k=0 \quad = \boxed{[illegible]}$$

$$= \frac{1}{\sqrt[5]{2}}\left[\cos\left[\frac{17\pi}{30}\right] - i\sin\left[\frac{17\pi}{30}\right]\right] \quad k=1 \quad \boxed{[illegible]}$$

$$= \frac{1}{\sqrt[5]{2}}\left[\cos\left[\frac{29\pi}{30}\right] - i\sin\left[\frac{29\pi}{30}\right]\right] \quad k=2 \quad \boxed{-.865 - i.091}$$

$$= \frac{1}{\sqrt[5]{2}}\left[\cos\left[\frac{41\pi}{30}\right] - i\sin\left[\frac{41\pi}{30}\right]\right] \quad k=3 \quad \boxed{-.354 + i.795}$$

$$= \frac{1}{\sqrt[5]{2}}\left[\cos\left[\frac{53\pi}{30}\right] - i\sin\left[\frac{53\pi}{30}\right]\right] \quad k=4 \quad \boxed{.647 + i.583}$$

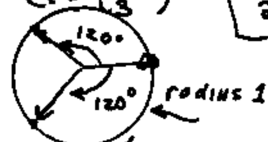
15] $1^{1/3}$ has 3 values 1, $\text{cis}\left(\frac{2\pi}{3}\right)$, $\text{cis}\left(\frac{4\pi}{3}\right)$

$1^{-1/3}$ has 3 values 1, $\text{cis}\left(-\frac{2\pi}{3}\right)$, $\text{cis}\left(-\frac{4\pi}{3}\right)$

multiplying these together we have 9 possibilities of which 3 are numerically distinct

The answers are $\boxed{1}$ $\quad \text{cis}\left(\frac{2\pi}{3}\right) = \boxed{-\frac{1}{2} + i\frac{\sqrt{3}}{2}}$

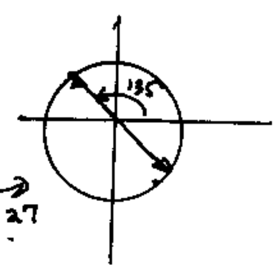
$\text{cis}\left[\frac{4\pi}{3}\right] = \boxed{-\frac{1}{2} - i\frac{\sqrt{3}}{2}}$



16] $(9i)^{3/2} = \left[\sqrt[2]{9}\right]^3 \angle \frac{\pi}{2}\times\frac{3}{2} + \frac{(2k\pi)(3)}{2} \qquad k=0,1$

$$= 27\angle\frac{3\pi}{4} + 3k\pi \qquad k=0,1$$

$$= \pm\left[-19.09 + i\,19.09\right]$$

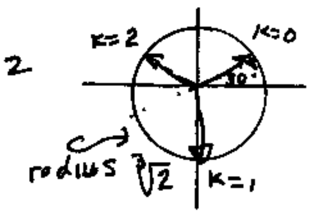


17]

$(1+i)^{6/2} = (1+i)^3 = (2i)(1+i) = \boxed{-2+2i}$

18] $(1+i)^{4/6} = (1+i)^{2/3} = \sqrt[3]{2} \angle \frac{2}{3}\frac{\pi}{4} + 2k\pi \cdot \frac{2}{3}$

$= \sqrt[3]{2} \angle \frac{\pi}{6} + \frac{4k\pi}{3}$ 0, 1, 2



$\boxed{1.09 + i.63}$

$\boxed{-i\, 1.26}$

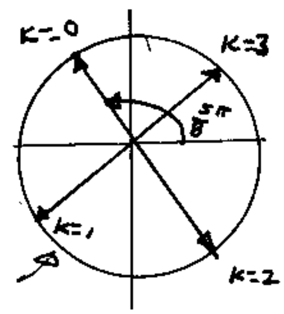
$\boxed{-1.09 + i.63}$

19] $(64i)^{10/8} = (64i)^{5/4} = \left[\sqrt[4]{64}\right]^5 \angle \frac{\pi}{2}\cdot\frac{5}{4} + 2k\pi\frac{5}{4}$

$= 128\sqrt{2} \angle \frac{5\pi}{8} + \frac{5k\pi}{2}$ k=0, 1, 2, 3

$= \boxed{\pm\left[-69.27 + i\,167.2\right]}$ k=0, 2

$= \boxed{\pm\left[-167.2 - i\,69.27\right]}$ k=1, 3

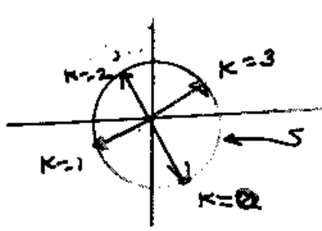


20] $(1+i)^{-5/4} = \left[\sqrt[8]{2}\right]^{-5} \angle \frac{\pi}{4}\left(\frac{-5}{4}\right) + 2k\pi\left(\frac{-5}{4}\right)$

$= \frac{1}{\left[\sqrt[8]{2}\right]^5} \angle -\frac{5\pi}{16} - \frac{k\pi 5}{2}$ k=0, 1, 2, 3

$= \boxed{\pm(.3602 - i.539)}$ k=0, k=2

$\boxed{\pm\left[.539 + i.36\right]}$ $\boxed{k=1, 3}$



radius $\left[\frac{1}{\sqrt[8]{2}}\right]^5$

21 We will choose $k=2$. Thus $w \doteq 1.682 \angle \frac{21\pi}{8}$

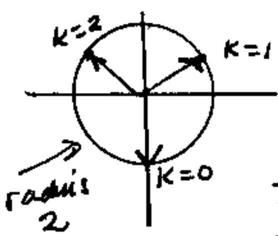
We will raise this to $\frac{4}{3}$ power. Thus: $\left[1.682 \angle \frac{21\pi}{8}\right]^{4/3}$

$= \left[\sqrt[3]{[1.682]}\right]^4 \angle \frac{4}{3}\,\frac{21\pi}{8} + \frac{4}{3} * 2k\pi \qquad k=0,1,2$

$= 2 \angle \frac{7\pi}{2} + \frac{4}{3} * 2k\pi = 2 \angle \left(3\tfrac{1}{2}\right)\pi + \frac{8k\pi}{3} \qquad k=0,1,2$ subtract 4π

$= 2 \angle \frac{-\pi}{2} + 2k\pi + \frac{2}{3}k\pi = 2 \angle -\frac{\pi}{2} + \frac{2}{3}k\pi$ roots plotted below

subtract $2k\pi$



When $k=0$, the root $1.682 \angle \frac{21\pi}{8}$ when raised to the $\frac{4}{3}$ power yields $-i2$. Thus $w^{4/3} + 2i = 0$ will be satisfied.

$\boxed{22}$ $az^2+bz+c=0$, divide by a get

$z^2 + \frac{b}{a}z + \frac{c}{a} = 0$. Notice that $\left(z+\frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} = z^2 + \frac{b}{a}z$

use this here

$\left(z+\frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$ get: $\left(z+\frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2-4ac}{4a^2}$

$\left(z+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$. Take sq. root

both sides: $z + \frac{b}{2a} = \left(\frac{b^2-4ac}{4a^2}\right)^{1/2}$

$z = \frac{-b}{2a} + \left(\frac{b^2-4ac}{4a^2}\right)^{1/2} = \frac{-b}{2a} + \frac{[b^2-4ac]^{1/2}}{2a}$

this has in general 2 values that are negatives of each other.

Note: if a, b, c. not real we do not necessarily get real roots or a pair of roots that are conjugates of each other. Suppose $a=1$, $b=i$, $c=0$

Equation is $z^2 + iz = 0$. Roots are $z=0$, $z=-i$

These are not conjugates.

23] $w^2 + w + i/4 = 0$

$w = \dfrac{-1 + (1-i)^{1/2}}{2}$

$w = -\frac{1}{2} \pm \frac{1}{2}\sqrt[4]{2} \angle -\frac{\pi}{8} =$ plus sign $\boxed{.0493 - .2275i}$ neg sign $\boxed{-1.0493 + .2275i}$

24] $w^2 + iw + 1 = 0$

$w = \dfrac{-i + (-1-4)^{1/2}}{2}$ $\qquad w = -\frac{i}{2} \pm \frac{i\sqrt{5}}{2}$

$\boxed{w = \frac{i}{2}\left[-1 + \sqrt{5}\right]}$, $\boxed{w = \frac{i}{2}\left[-1 - \sqrt{5}\right]}$

25] $w^4 + w^2 + 1 = 0$, $w^2 = \dfrac{-1 + (-3)^{1/2}}{2}$

$w^2 = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$ $\qquad w = \left(-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right)^{1/2}$

$w = \left[1 \angle \pm 2\pi/3\right]^{1/2}$ $\qquad w = 1 \angle \pi/3 = \boxed{\frac{1}{2} + i\frac{\sqrt{3}}{2}}$

w: $\boxed{-\frac{1}{2} - i\frac{\sqrt{3}}{2}}$ $\quad \boxed{-\frac{1}{2} + i\frac{\sqrt{3}}{2}}$ $\quad \boxed{\frac{1}{2} - i\frac{\sqrt{3}}{2}}$

26] $w^6 + w^3 + 1 = 0$ $\qquad (w^3)^2 + (w^3)^1 + 1 = 0$

$w^3 = \dfrac{-1 + (1-4)^{1/2}}{2}$ $\qquad w^3 = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$

$w^3 = 1 \angle \pm \frac{2\pi}{3}$

$w = 1 \angle \frac{2\pi}{9} + \frac{2k\pi}{9}$ $\quad k = 0, 1, 2$

$\boxed{\begin{array}{l} .766 + i\,.6428 \\ .1736 + i\,.9848 \\ -.9397 + i\,.342 \end{array}}$

$w = 1 \angle -\frac{2\pi}{9} + \frac{2k\pi}{9}$ $\quad k = 0, 1, 2$

$\boxed{\begin{array}{l} .766 - i\,.6428 \\ .1736 - i\,.9848 \\ -.9397 - i\,.342 \end{array}}$

27| a) $(z-1)(z^n + z^{n-1} + \dots z + 1) = z^{n+1} + z^n + \dots z$

$- z^n - z^{n-1} \dots - 1 = (z^{n+1} - 1)$

b) $z^4 + z^3 + \dots z + 1 = \dfrac{z^5 - 1}{z - 1}$ for $z \neq 1$

Note $z = 1$ is not a root of the given equation.

The roots of $z^4 + z^3 + z^2 + z + 1 = 0$ are the roots of $z^5 - 1 = 0$ $z = 1^{1/5}$

$$\boxed{z = \operatorname{cis}\left[\frac{2k\pi}{5}\right] \quad k = 1, 2, 3, 4}$$

28| a) $z^{1/n} = \sqrt[n]{r} \angle \frac{\theta}{n} + \frac{2k\pi}{n}$ $k = 0, 1, 2 \dots n-1$

Sum of roots $\displaystyle\sum_{k=0}^{n-1} \sqrt[n]{r} \operatorname{cis}\left(\frac{\theta}{n}\right) \operatorname{cis}\left(\frac{2k\pi}{n}\right)$

b) Sum of roots $= \sqrt[n]{r} \operatorname{cis}\left(\frac{\theta}{n}\right) \displaystyle\sum_{k=0}^{n-1} \left[\operatorname{cis}\left(\frac{2\pi}{n}\right)\right]^k$

$= \sqrt[n]{n} \operatorname{cis}\left(\frac{\theta}{n}\right) \dfrac{\left[\operatorname{cis}\left(\frac{2\pi}{n}\right)\right]^{n} - 1}{\operatorname{cis}\left(\frac{2\pi}{n}\right)} = \sqrt[n]{r} \operatorname{cis}\left(\frac{\theta}{n}\right) \dfrac{\operatorname{cis}(2\pi) - 1}{\operatorname{cis}\left(\frac{2\pi}{n}\right)}$

$= \boxed{0}$

29| a) $z = r\angle\theta$, $z^{1/2} = \pm\sqrt{r} \angle \frac{\theta}{2} =$

$\pm\sqrt{r}\left[\cos\frac{\theta}{2} + i \sin\frac{\theta}{2}\right]$. Now if $0 \leq \theta \leq \pi$

arg $z^{1/2}$ is either in 1st quadrant or 3rd quad.

If $\sin\theta/2$ is positive then $\cos\frac{\theta}{2}$ is positive

or if $\sin\frac{\theta}{2}$ is negative then $\cos\frac{\theta}{2}$ is negative

Thus $z^{1/2} = \pm\sqrt{r}\left[\sqrt{\frac{1+\cos\theta}{2}} + i\sqrt{\frac{1-\cos\theta}{2}}\right]$

b) If $-\pi < \theta < 0$, then arg $z^{1/2}$ is either in the 4th quadrant or the 2nd quadrant. Thus if $\sin[\theta/2]$ is positive $\cos\theta/2$ is neg. or vice-versa

Thus $z^{1/2} = \pm\sqrt{r}\left[-\sqrt{\frac{1+\cos\theta}{2}} + i\sqrt{\frac{1-\cos\theta}{2}}\right]$

29) continued (c)

$r=2 \quad \theta=\pi/6$

$$z^{1/2} = \pm\sqrt{2}\left[\sqrt{\frac{1+\cos(\pi/6)}{2}} + i\sqrt{\frac{1-\cos(\frac{\pi}{6})}{2}}\right]$$

$=$

$$= \pm[1.366 + i\,.366]$$

$r=2,\ \theta=-\pi/6 \qquad z^{1/2} = \pm\sqrt{2}\left[-\sqrt{\frac{1+\cos(\frac{\pi}{6})}{2}} + i\sqrt{\frac{1-\cos(\frac{\pi}{6})}{2}}\right]$

$$= \pm\left[-1.366 + i\,.366\right]$$

30) a) $(a+ib) = (x+iy)^{1/2}$. Square both sides $a^2-b^2+i2ab = x+iy$
Equate corresponding parts: $x = a^2-b^2$, $y = 2ab$.

b) Now if $y \neq 0$ it follows (since $y=2ab$) that $a \neq 0$.
Thus $b = y/(2a)$. Eliminating b from $x=a^2-b^2$ we have: $x = a^2 - \frac{y^2}{4a^2}$ or $4a^2x = 4a^4 - y^2$ or $a^4 - a^2x - \frac{y^2}{4} = 0$, Using the quadratic formula we get $a^2 = \frac{x \pm \sqrt{x^2+y^2}}{2}$. The minus sign must be rejected. Recall that $y^2 \neq 0$, Using the minus sign would require that $a^2 < 0$ which not possible since a is real.

(c) From (1) have $b^2 = a^2 - x$. Using result of (a) now have: $b^2 = \frac{x+\sqrt{x^2+y^2}}{2} - \frac{2x}{2} = \frac{-x+\sqrt{x^2+y^2}}{2}$

d) $y = 2ab$. If $y>0$, a and b are of like sign. Thus if choose plus sign in 5) You'd better choose plus sign in 6). If choose minus sign in 5) must choose minus sign in 6).

e) Since $y = 2ab$, if $y<0$, then a and b are of opposite sign. Thus a plus sign in 5) means using a minus sign in 6) and vice-versa.

f) If $y=0$ then (1) and (2) become $x = a^2-b^2$, $0 = 2ab$. Thus at least one of the variables a and b must be zero. If $x>0$, then $x=a^2-b^2$ is satisfied if and only if $b=0$. Thus $x=a^2$, $a = \pm\sqrt{x}$. If $x<0$, then $x=a^2-b^2$ is satisfied if and only if $a=0$. Thus $x=-b^2$, $b^2=-x$, $b = \pm\sqrt{-x}$. If $x=0$, then $x=a^2-b^2$ requires $a=b=0$.

g) Need $(i)^{1/2}$. Take $x=0, y=1$. Thus $a = \pm\frac{1}{\sqrt{2}}$, $b=\pm\frac{1}{\sqrt{2}}$ from (5) and (6). Since $y>0$, a and b are of like sign. Thus $i^{1/2} = \frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}$ or $i^{1/2} = \frac{-1}{2} - \frac{i}{\sqrt{2}}$.

31) a) Let $z = r\,\text{cis}(\theta)$, Assume $m \ge 1$. Then

$$z^{1/m} = \sqrt[m]{r}\ \text{cis}\left(\frac{\theta}{m}+\frac{2k\pi}{m}\right),\quad z^{-1/m} = \frac{1}{\sqrt[m]{r}}\ \text{cis}\left[\frac{-\theta}{m}+\frac{2k'\pi}{-m}\right]$$

$k=0,1,2,\dots m-1$ $\qquad k'=0,1,\dots m-1$

$$z^{1/m}z^{-1/m} = \frac{\sqrt[m]{r}}{\sqrt[m]{r}}\ \text{cis}\left[\frac{\theta}{m}+\frac{2k\pi}{m}\right]\text{cis}\left[\frac{-\theta}{m}+\frac{2k'\pi}{-m}\right] =$$

$$= \text{cis}\left[\frac{2\pi}{m}\left[k-k'\right]\right]$$

$k=0,1,\dots m-1$
$k'=0,1,\dots m-1$

The preceding is not necessarily one.

If $m \le -1$, the same argument applies except that k and k' both range from 0 to $|m|-1$. Thus $z^{1/m}z^{-1/m}$ does not have to equal one if the roots are chosen randomly.

b) Yes, we can always find roots $z^{1/m}$ and $z^{-1/m}$ such that $z^{1/m}z^{-1/m} = 1$. In the expression $\text{cis}\left[\frac{2\pi}{m}(k-k')\right]$ we choose $k=k'$.

(a) Let $z = r\,\text{cis}(\theta)$. First compute $(z^n)^{1/m} =$
$[r^n \angle n\theta]^{1/m} = \left[\sqrt[m]{r}\right]^n \text{cis}\left[\frac{n\theta}{m} + \frac{2k\pi}{m}\right]$ $k=0,1,\cdots m-1$
$= \left[\sqrt[m]{r}\right]^n \text{cis}\left[\frac{n\theta}{m}\right] \text{cis}\left[\frac{2k\pi}{m}\right]$ $k=0,1,\cdots m-1$. Thus
$[z^n]^{1/m} = \left[\sqrt[m]{r}\right]^n \text{cis}\left[\frac{n\theta}{m}\right] \text{cis}\left[\frac{2k\pi}{m}\right]$, $k=0,1,\cdots m-1$ [1]
Now compute: $\left[z^{\frac{1}{m}}\right]^n = \left[\sqrt[m]{r}\ \text{cis}\left[\frac{\theta}{m} + \frac{2k'\pi}{m}\right]\right]^n$ $k'=0,\cdots m-1$
$= \sqrt[m]{r}\ \text{cis}\left[\frac{n\theta}{m} + \frac{2k'\pi n}{m}\right]$, $k'=0,\cdots m-1$. Thus:
$\left[z^{\frac{1}{m}}\right]^n = \left(\sqrt[m]{r}\right)^n \text{cis}\left(\frac{n\theta}{m}\right) \text{cis}\left[\frac{2k'\pi n}{m}\right]$ $k'=0,1,\cdots m-1$ [2]
To prove that results [1] and [2] are identical sets of complex numbers, we need prove only that the set of numbers $\text{cis}\left[\frac{2k\pi}{m}\right]$ $k=0,1,\cdots m-1$ is identical to the set of numbers $\text{cis}\left[\frac{2k'n\pi}{m}\right]$ $k'=0,1,\cdots, m-1$
The set of values of $\text{cis}\left(\frac{2k\pi}{m}\right)$ $k=0,\cdots m-1$ are all obviously distinct (no two alike) since as we know they are uniformly spaced around the unit circle in the complex plane, with spacing of $\frac{2\pi}{m}$ between values. This is easily proved and is not given here. Let us prove that $\text{cis}\left(\frac{2k'n\pi}{m}\right)$, $k'=0,\cdots(m-1)$ is a set of m distinct values. Suppose there are two values that are the same, i.e. $\text{cis}\left(\frac{2k'n\pi}{m}\right) = \text{cis}\left[\frac{2k''n\pi}{m}\right]$

We assume $\text{cis}\left(\frac{2k'n\pi}{m}\right) = \text{cis}\left[\frac{2k''n\pi}{m}\right]$ where $k' > k''$ and $0 \le k' \le m-1$, $0 \le k'' \le m-1$. This implies that $\frac{2k'n\pi}{m} - \frac{2k''n\pi}{m} = 2\pi L$ for some integer L, that is: $\frac{n(k'-k'')}{m} = L$
or $\frac{n}{m} = \frac{L}{k'-k''}$. Now notice that $k'-k'' \le m-1$. The equation
$\frac{n}{m} = \frac{L}{k'-k''}$ is a contradiction since we assumed that $\frac{n}{m}$ is an irreducible fraction. Thus our assumption that $\text{cis}\left(\frac{2k'n\pi}{m}\right) = \text{cis}\left(\frac{2k''n\pi}{m}\right)$ $k' \ne k''$, $0 \le k' \le m-1$, $0 \le k'' \le m-1$ must be false.
Therefore the values of $\text{cis}\left(\frac{2k'n\pi}{m}\right)$, $k'=0,1,\cdots m-1$ must be numerically distinct (no two alike). We now prove that for each value of k', $0 \le k' \le m-1$ there is just one value of k, $0 \le k \le m-1$ such that $\text{cis}\left(\frac{2k\pi}{m}\right) = \text{cis}\left[\frac{2k'\pi n}{m}\right]$. Suppose we are given k', then the following shows how to find k. We require that $\frac{2k\pi}{m} = \frac{2k'\pi n}{m} - 2\pi L$ where L is some integer and k satisfies $0 \le k \le m-1$. Thus $\frac{k}{m} = \frac{k'n}{m} - L$. Now if $\frac{k'n}{m}$ is an integer,

(continued, next pg.)

32 (a) cont'd

we take $L = \frac{k'n}{m}$ and $k=0$ to satisfy this equation. If $\frac{k'n}{m}$ is not an integer, it must be of the form: $\frac{k'n}{m} = I + \frac{P}{m}$ where I is an integer and $1 \le P \le m-1$. Now take $L = I$ and the equation $\frac{k}{m} = \frac{k'n}{m} - L$ becomes $\frac{k}{m} = \frac{P}{m}$. We thus take $k = P$, where k will satisfy $0 \le k \le m-1$. Thus we have found the value of k corresponding to k'. In summary, the set of m different values of $\text{cis}\left(\frac{2k\pi}{m}\right)$ is identical to the set of values of $\text{cis}\left(\frac{2k'\pi n}{m}\right)$, $k=0,\cdots m-1$, $k'=0,\cdots m-1$. Thus the sets of values generated in equations [1] and [2] [previous page] must be identical, [q.e.d].

32](b) $1^{1/4} = \text{cis}\left[\frac{2k\pi}{4}\right]$, $k=0,1,2,3$, $\left[1^{1/4}\right]^2 = \left[\text{cis}\left[\frac{2k\pi}{4}\right]\right]^2$
$= \text{cis}\left[k\pi\right]$ $k=0,1,2,3$ $= 1$ $(k=0)$, -1 $[k=1]$, 1 $[k=2]$, -1, $[k=3]$.
Thus $\left[1^{1/4}\right]^2 = \pm 1$. Now consider $(1^2)^{1/4} = 1^{\frac{1}{4}} =$ $\text{cis}\left(\frac{2k\pi}{4}\right)$ $(k=0,1,2,3)$ which equals 1 if $k=0$, i if $k=1$, -1 if $k=2$, $-i$ if $k=3$. Thus $(1^2)^{1/4} =$ ± 1 and $\pm i$ [four different values,] while $\left(1^{1/4}\right)^2 = \pm 1$ (two different values).

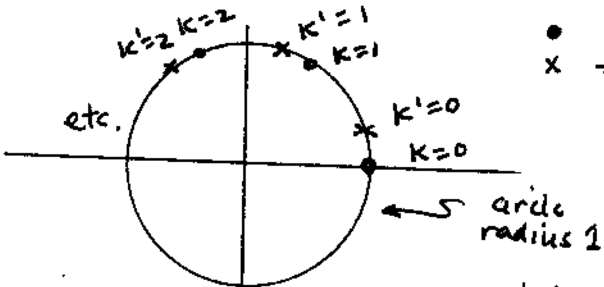
33] next pg.

Sec. 1.4

33| a)

$1^{\frac{1}{m}} = \text{cis}\left(\frac{2k\pi}{m}\right) \quad k=0,1,2\cdots m-1, \quad i^{1/m} = \text{cis}\left(\frac{\pi}{2m} + \frac{2k'\pi}{m}\right)$ $k'=0,1,\ldots m-1$

$i^{1/m} = \text{cis}\left(\frac{\pi}{2m}\right)\text{cis}\left(\frac{2k'\pi}{m}\right) \quad k'=0,1,\cdots m-1$



$\bullet$ = value of $1^{\frac{1}{m}}$

x = value of $i^{1/m}$

From the diagram, we see that by selecting $k=k'$ we have minimized $\left|1^{\frac{1}{m}} - i^{\frac{1}{m}}\right|$. Thus with this choice

$$\left|1^{\frac{1}{m}} - i^{\frac{1}{m}}\right| = \left|\text{cis}\left(\frac{2\pi k}{m}\right) - \text{cis}\left(\frac{\pi}{2m}\right)\text{cis}\left(\frac{2\pi k'}{m}\right)\right| =$$

$$\left|\left(\text{cis}\left(\frac{2\pi k}{m}\right)\right)\left(1 - \text{cis}\left(\frac{\pi}{2m}\right)\right)\right| = \left|\text{cis}\left(\frac{2\pi k}{m}\right)\right|\left|1-\text{cis}\left(\frac{\pi}{2m}\right)\right|$$

$$= \left|1-\text{cis}\left(\frac{\pi}{2m}\right)\right| = \left|1-\cos\left(\frac{\pi}{2m}\right) - i\sin\left(\frac{\pi}{2m}\right)\right| =$$

$$\sqrt{\left[1-\cos\left(\frac{\pi}{2m}\right)\right]^2 + \sin^2\left(\frac{\pi}{2m}\right)} = \sqrt{1-2\cos\left(\frac{\pi}{2m}\right)+\cos^2\left(\frac{\pi}{2m}\right)+\sin^2\left(\frac{\pi}{2m}\right)}$$

$$= \sqrt{2\left(1-\cos\left(\frac{\pi}{2m}\right)\right)} = \sqrt{2 \cdot 2\sin^2\frac{\pi}{4m}} = 2\sin\left(\frac{\pi}{4m}\right)$$

b) Refer to the figure of part (a). The magnitude of the vector for $1^{\frac{1}{m}} + i^{1/m}$ will be maximized if the angle between the vectors for $1^{1/m}$ and $i^{1/m}$ is minimized. This means we should use the same value for k and k' in the expressions for $1^{1/m}$ and $i^{1/m}$ given above.

Thus $\left|1^{\frac{1}{m}} + i^{1/m}\right| = \left|\text{cis}\left(\frac{2\pi k}{m}\right) + \text{cis}\left(\frac{\pi}{2m}\right)\text{cis}\left(\frac{2\pi k'}{m}\right)\right|$

$$= \left|\text{cis}\left(\frac{2\pi k}{m}\right)\right|\left|1+\text{cis}\left(\frac{\pi}{2m}\right)\right| = \left|1+\text{cis}\left(\frac{\pi}{2m}\right)\right| = \sqrt{\left[1+\cos\left(\frac{\pi}{2m}\right)\right]^2+\sin^2\left(\frac{\pi}{2m}\right)}$$

$$= \sqrt{1+\cos^2\frac{\pi}{2m}+2\cos\frac{\pi}{2m}+\sin^2\frac{\pi}{2m}} = \sqrt{(2)\left[1+\cos\frac{\pi}{2m}\right]} = \sqrt{2 \cdot 2\cos^2\frac{\pi}{4m}} = \boxed{2\cos\frac{\pi}{4m}}$$

34]

Let $z = \text{cis}\,\theta$ in problem 27 formula

Then: $1 + \text{cis}\,\theta + (\text{cis}\,\theta)^2 \ldots (\text{cis}\,\theta)^n = \dfrac{1 - [\text{cis}\,\theta]^{n+1}}{1 - \text{cis}\,\theta}$

Now use DeMoivre's Thm

$1 + \text{cis}\,\theta + \text{cis}(2\theta) + \text{cis}(3\theta) \cdots + \text{cis}(n\theta) = \dfrac{1 - \text{cis}[(n+1)\theta]}{1 - \text{cis}\,\theta}$ [ ]

$= \dfrac{\text{cis}(n+1)\theta - 1}{\text{cis}(\theta) - 1} = \dfrac{\text{cis}[n+1]\frac{\theta}{2}}{\text{cis}\,\frac{\theta}{2}} \left[\dfrac{\text{cis}\left[\frac{(n+1)\theta}{2}\right] - \text{cis}\left[-\frac{(n+1)\theta}{2}\right]}{\text{cis}(\frac{\theta}{2}) - \text{cis}(-\frac{\theta}{2})} \right]$

[Note that $\text{cis}\,\psi - \text{cis}(-\psi) = 2i\sin\psi$, ψ any real.]

Thus $1 + \text{cis}\,\theta + \text{cis}(2\theta) \cdots \text{cis}(n\theta) = \text{cis}\left(\frac{n\theta}{2}\right)\left[\dfrac{\sin(n+1)\theta/2}{\sin(\theta/2)}\right] =$

$\left[\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2}\right]\left[\dfrac{\sin(n+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right] = 1 + \cos\theta + i\sin\theta + \cos 2\theta + i\sin 2\theta \ldots \cos n\theta + i\sin(n\theta)$

Now equate reals on each side of the preceding equation, and imaginaries:

$\left(\cos\frac{n\theta}{2}\right)\left[\dfrac{\sin(n+1)\theta/2}{\sin\theta/2}\right] = 1 + \cos\theta + \cos 2\theta \ldots \cos n\theta.$

$\sin\frac{n\theta}{2}\,\dfrac{\sin[(n+1)\theta/2]}{\sin(\theta/2)} = \sin\theta + \sin 2\theta + \sin 3\theta \ldots \sin n\theta$ q.e.d.

35]

from problem 27 have $1 + p + p^2 + \cdots p^n = \dfrac{1 - p^{n+1}}{1 - p}$
where $p = z$.

or $1 + p + p^2 + \cdots p^{n-1} = \dfrac{1 - p^n}{1 - p}$. Take $p = \text{cis}\left(\frac{2\pi}{n}\right)$

$1 + \text{cis}\left(\frac{2\pi}{n}\right) + \text{cis}\left(\frac{4\pi}{n}\right) + \cdots \text{cis}\left[\frac{(2\pi)(n-1)}{n}\right] = \dfrac{1 - \text{cis}\left(\frac{2\pi n}{n}\right)}{1 - \text{cis}\left(\frac{2\pi}{n}\right)} = 0$

Since $\text{cis}(2\pi) = 1$. Thus:

$1 + \text{cis}\left(\frac{2\pi}{n}\right) + \text{cis}\left[\frac{4\pi}{n}\right] + \cdots \text{cis}\left[\frac{(2\pi)(n-1)}{n}\right] = 0$

Break preceding into reals and imaginaries:

$1 + \cos\left[\frac{2\pi}{n}\right] + \cos\left[\frac{4\pi}{n}\right] + \cos\left[\frac{8\pi}{n}\right] \cdots \cos\left[\frac{2\pi(n-1)}{n}\right] = 0$ [reals]

or $\left.\begin{aligned} &\cos\left[\frac{2\pi}{n}\right] + \cos\left[\frac{4\pi}{n}\right] \ldots \cos\left[\frac{2\pi(n-1)}{n}\right] = -1 \\ &\sin\left(\frac{2\pi}{n}\right) + \sin\left(\frac{4\pi}{n}\right) \cdots \sin\left[\frac{(2\pi)(n-1)}{n}\right] = 0 \end{aligned}\right\}$ q.e.d.

36) Matlab code

```
W = [1 1 1 1 1 1]
roots (W)
ans =  .5 + .866i
       .5 - .866i
       -1
       -.5 + .866i
       -.5 - .866i .
```

Can use $z^6 - 1 = 0$, $z = 1^{1/6}$

$= \text{cis}\left[\frac{2k\pi}{6}\right]$ $k = 1, 2, 3, 4, 5$ but not $k = 0$

$= \text{cis}\left[\frac{\pi}{3}\right] = .5 + .866i$

$= \text{cis}\left[\frac{2\pi}{3}\right] = -.5 + .866i$

$= \text{cis}\left[\pi\right] = -1$

$\text{cis}\left[\frac{4\pi}{3}\right] = -.5 - .866i$

$\text{cis}\left[\frac{5\pi}{3}\right] = .5 - .866i$

37) (a) With MATLAB $z^{n/m} = \left[\sqrt[m]{r}\right]^n \angle \theta \frac{n}{m}$

where $-\pi < \theta \le \pi$ [MATLAB uses princ argument]

Thus $z^{n/m} = \left[\sqrt[m]{r}\right]^n \left[\cos\left[\theta \frac{n}{m}\right] + i \sin\left[\theta \frac{n}{m}\right]\right]$

MATLAB: $i^{3/4} = .3827 + i\,.9239$

$(.3827 + i\,.9239)^{4/3} = i$ as expected.

Why: $i^{3/4} = \cos\left[\frac{\pi}{2}\frac{3}{4}\right] + i \sin\left[\frac{\pi}{2}\frac{3}{4}\right] = \text{cis}\left[\frac{3\pi}{8}\right]$

raise preceding to $\frac{4}{3}$. Note $-\pi < \frac{3\pi}{8} \le \pi$

$\therefore \left[i^{3/4}\right]^{4/3} = \left[\cos\left[\frac{3\pi}{8}\right] + i \sin\left[\frac{3\pi}{8}\right]\right]^{4/3}$

$= \cos\frac{\pi}{2} + i \sin \pi/2 = \boxed{i}$

Sec 1.4 continued.

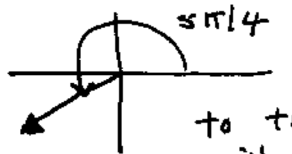
37] (b) FROM MATLAB: $i^{5/2} = -.707 - i.707$

with Matlab: $\left(i^{5/2}\right)^{2/5} = (-.707 - i.707)^{2/5} =$

$.5878 - i.809$ which $\neq i$

Why? $i^{5/2} = \text{cis}\left[\frac{\pi}{2}\,\frac{5}{2}\right] + i\sin\left[\frac{\pi}{2}\,\frac{5}{2}\right]$

$= \text{cis}\left[\frac{5\pi}{4}\right] + i\sin\left[\frac{5\pi}{4}\right] = -.707 - i.707$



. To raise the preceding to the 2/5 power, MATLAB will use princ. arg.

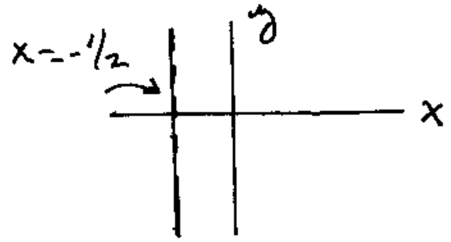
Thus $(-.707 - i.707)^{2/5} = 1\angle\frac{-3\pi}{4}\cdot\frac{2}{5} = \text{cis}\left[-.3\pi\right]$

$= \boxed{.5878 - i.809}$

We did not recover i because in computing $(-.707 - i.707)^{2/5}$ Matlab used the princ argument and not $\theta = \frac{5\pi}{4}$. Had it used $\frac{5\pi}{4}$, i would have been recovered.

Sec 1.5

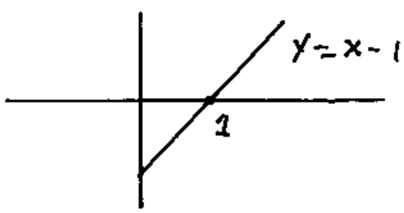
1) $\operatorname{Re} z = -1/2, \quad x = -\frac{1}{2}$



The solution is the line $x = -1/2$

2) $\operatorname{Re}(z) = \operatorname{Im}(z+i) \qquad z = x + iy$

$x = y + 1 \qquad y = x - 1$



3) $\operatorname{Re}(z) > \operatorname{Im}(z+i)$

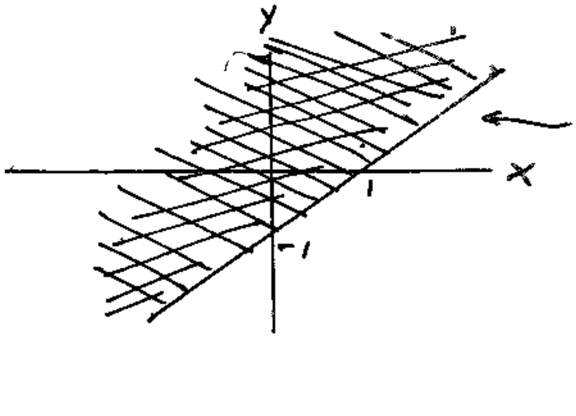
$x > y + 1 \qquad y < x - 1$

$y < x-1$ is the shaded area below the line $y = x-1$ but not including the line

4) $\operatorname{Re}(z) \leq \operatorname{Im}(z+i) \qquad x \leq y+1$

$y \geq x - 1$

The area on the line and the shaded area above it

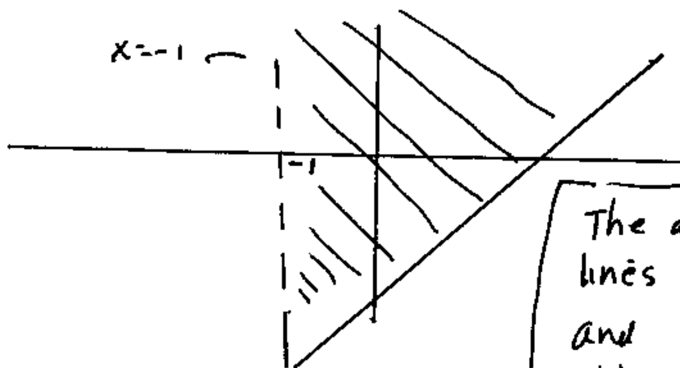


The line $y + 1 = x$

section 1.5

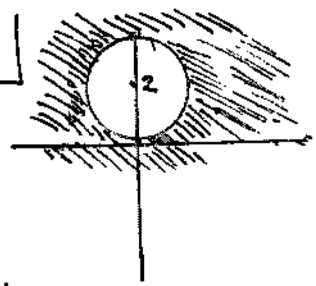
5 $\quad -1 < Re(z) \leq Im(z+i)$

$-1 < x \leq y+1$



The area between the lines $y+1=x$ and $x=-1$, not including points on $x=-1$ but including those on $y+1=x$

6



The points on and outside this circle.

7 $\quad z\bar{z} = x^2+y^2 = 1+i$ Since left side is real, this has $\boxed{\text{no solution}}$

8 $\quad x^2+y^2 = x \qquad x^2-x+y^2=0$

$(x-\frac{1}{2})^2+y^2 = 1/4 = (\frac{1}{2})^2$ Circle of radius $\frac{1}{2}$

centered at $(\frac{1}{2},0)$.

9 $\quad Re(z) = Im(z^2) \qquad x = Im[x^2-y^2+i2xy]$

$x=2xy$ satisfied if $x=0$, suppose $x\neq 0$, Divided by x: $1=2y$, $y=1/2$

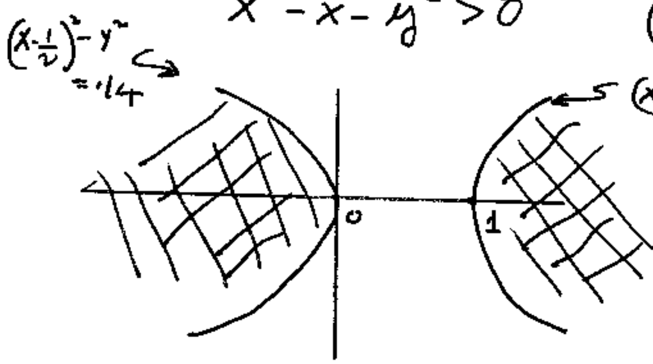
Answers $\boxed{x=0,\ -\infty < y < \infty}$ or $\boxed{y=\frac{1}{2}\quad |x|>0}$

10)

$$Re(z) < Re(z^2) = Re\left[x^2 - y^2 + i\,2xy\right]$$

$$x < x^2 - y^2$$

$$x^2 - x - y^2 > 0 \qquad \left(x - \tfrac{1}{2}\right)^2 - y^2 > \tfrac{1}{4}$$



Solution is shaded area bounded by this set of hyperbolas but not including the hyperbolas themselves

11) Suppose $e^{|z|} = 1$ $\quad |z| = 0$

suppose $e^{|z|} = 2$, $\quad |z| = \text{Log}\, 2$

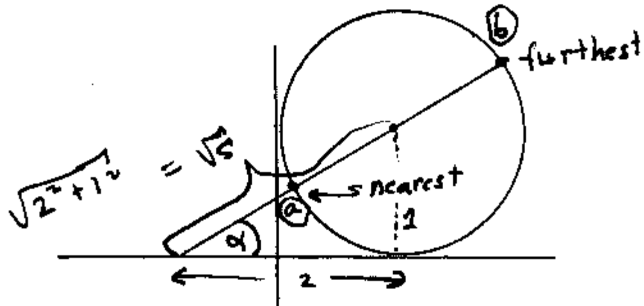
Answer is disc centered at origin with center punched out. Outer radius is Log 2. Circumferce $|z| = \text{Log}2$ is included

12)

next page

sec 1.5

12



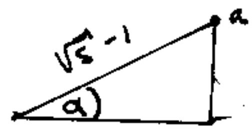
a is nearest distance which is $\boxed{\sqrt{5}-1}$

b is furthest distance which is $\boxed{\sqrt{5}+1.}$

Note $\cos\alpha = 2/\sqrt{5}$

$\sin\alpha = 1/\sqrt{5}$

To get coordinates of (a).



x coord $= -1 + (\sqrt{5}-1)\cos\alpha$

$= -1 + \dfrac{(\sqrt{5}-1)2}{\sqrt{5}} = 1 - \dfrac{2}{\sqrt{5}}$, y coordinate is $(\sqrt{5}-1)\sin\alpha = \dfrac{\sqrt{5}-1}{\sqrt{5}}$

Similarly to get x coordinate of (b): $-1 + (\sqrt{5}+1)\cos\alpha = 1 + \dfrac{2}{\sqrt{5}}$

and y coordinate $(\sqrt{5}+1)\sin\alpha = \dfrac{\sqrt{5}+1}{\sqrt{5}} = \dfrac{5+\sqrt{5}}{5}$

13 $\boxed{|z-i| < 1}$

14 $|z-(-1-2i)| > 3$

or $|z+1+2i| > 3$

15 $\boxed{0 < |z-2+i| \le 4}$

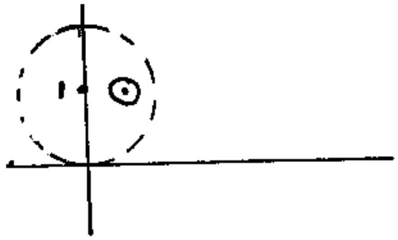
16 $1 < |z+1-3i| \le 4$

17 $\boxed{|z-1| + |z+1| = 2}$

18 next pg.

Sec 1.5

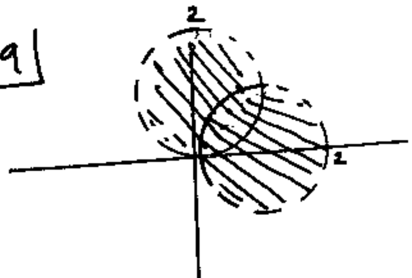
18) a)



CAN Use $|z-[.8+i]|<.1$ this not the only answer

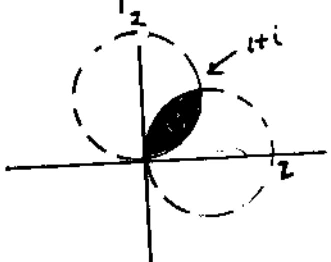
b) $0<|z-(.8+i)|<.1$

19)



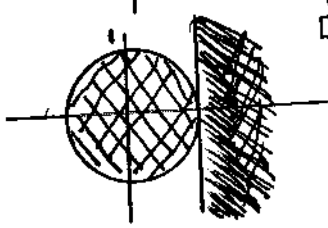
all the shaded area The boundaries are not included. This is a domain since open and connected.

20)



ans. is the shaded area, but the boundary points that lie on the arcs along $|z-i|=1$ and $|z-1|=1$ are not included. Is a domain. [open and connected]

21)



ans. is the shaded area and includes the points on the boundariés. Not a domain connected but not open.

22) There are no points common to both A and B. Answer is the null set. Not a domain.

23) $|z|>0$. The origin $\boxed{z=0}$ is a boundary point. It is $\boxed{\text{not in the set}}$

24)

Boundary points are the points on the circle $|z-i|=1$

They do not belong to the set.

Sec 1.5

25] $\frac{1}{3} < \frac{1}{|z-i|} \leq \frac{1}{2}$

implies $3 > |z-i| \geq 2$

Boundary points are on circle $|z-i| = 3$ and do not belong to the set and are on circle $|z-i| = 2$ and do belong

26] $\text{Log}|z| \geq 0$ implies that

$|z| \geq 1$. Boundary points are on circle $|z| = 1$ and do belong to the set

27] The elements of the set are
$i, ie^{i/2}, ie^{i/3}, ie^{i/4}, \ldots$ etc
Each element is a boundary point [draw a small circle around it]. However, $ie^0 = i$ is also a boundary point but does not belong to the set. Every neighborhood of i contains members of the set as well as points not belonging to the set.

28] This set $-1 \leq \text{Re}(z) \leq 5$ [or $-1 \leq x \leq 5$]
The boundary points are on the lines $x = -1$ and $x = 5$ $[-\infty < y < \infty]$. The set is closed because it contains all its boundary points.

29] The boundary points lie along the lines $x = -1$ $[-\infty < y < \infty]$ and $x = 5$ $[-\infty < y < \infty]$. The points on the line $x = 5$ do not belong to the set. $\therefore$ set is not closed

30| The set of points $ie, ie^{1/2}, ie^{1/3} \ldots$ has boundary point $ie^0 = i$ which does not belong to the set. $\therefore$ set is not closed

31| No, a boundary point is not necessarily an accumulation point. Consider for example the set consisting of just the one point $z=0$. This point is a boundary point but <u>not</u> an accumulation point. Any neighborhood of $z=0$ will not contain any elements of the set except $z=0$.

32| For problem 28, the accumulation points are all the points in the given set.

For problem 29, the accumulation points are all the points in the given set plus points on the line $\mathrm{Re}(z)=5$ [or $x=5$, $-\infty < y < \infty$]

For problem 30, the only accumulation point is at $z = i$

33|

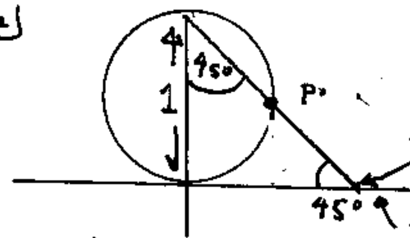
$$\sin\left(\frac{\pi}{x}\right)=0 \quad \therefore \quad \frac{\pi}{x} = n\pi, \quad x = \frac{1}{n}$$

n is an integer. We require $y=0$, $x=\frac{1}{n}$, $n=\pm 2, \pm 3, \pm 4 \ldots$ for a solution inside the unit circle. The points are on the real axis and cluster about $z=0$. The accumulation point is $z=0$. (It does not belong to the set since no finite value of n will produce $z=0$).

To prove that $z=0$ is an accum pt.: Consider the neighborhood $|z| < \epsilon$ where $\epsilon > 0$ is an arbitrary pos. number. Now consider a point for which $y=0$, and x is chosen such that $0 < x \le \frac{\epsilon}{2}$ and $\frac{1}{x}$ is an integer. This point belongs to the given set and lies in the given neighborhood of $z=0$. $\therefore$ any neighborhood of $z=0$ has a point in the set

34]

a]

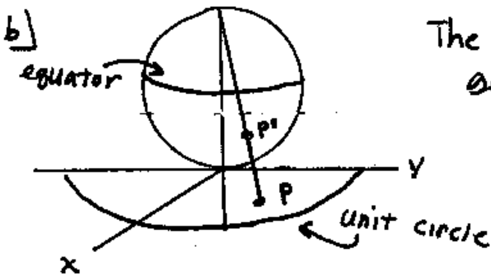


The point P gets projected onto a point P' lying on the equator of the sphere

a point on $|z|=1$

Thus the unit circle $|z| = 1$ gets projected onto the equator of the sphere.

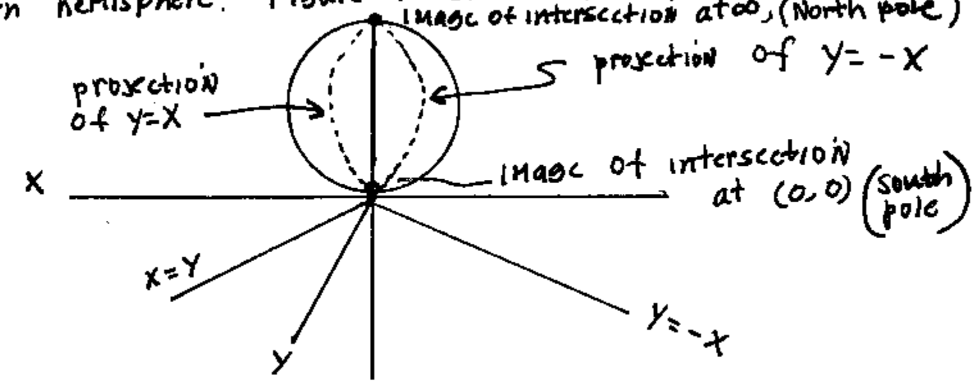
b]



The points inside the unit circle get projected onto southern (lower) hemisphere, not including the equator.

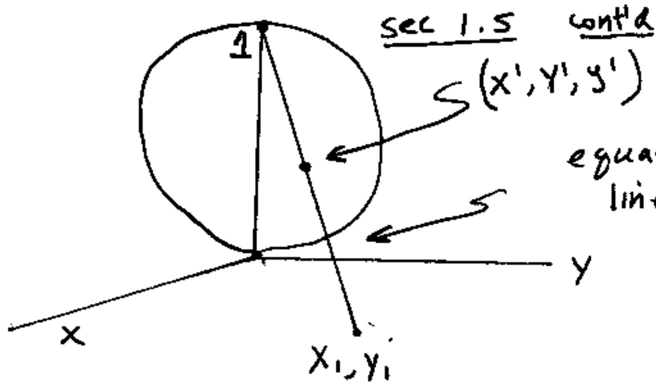
c] The points outside the unit circle get projected onto northern hemisphere. Figure is similar to the one above

35]



continued, next page

36|



equation of this line $\quad z = \dfrac{x_1 - x}{x_1}$ (1)

$z = \dfrac{y_1 - y}{y_1}$ (2)

a) Equation of the sphere: $x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$

Notice from (1) and (2) that $x = x_1[1 - z]$, $y = y_1[1 - z]$

Use this in the equation of the sphere. Get

$$x_1^2 (z-1)^2 + y_1^2 (z-1)^2 + \left(z - \tfrac{1}{2}\right)^2 = \tfrac{1}{4}$$

Put $x_1^2 + y_1^2 = r^2$. Thus $r^2 (z-1)^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$

or $z^2 [r^2 + 1] - z[1 + 2r^2] + r^2 = 0$

Using the quadratic formula, we solve the preceding equation for z. We get two roots

$z = \dfrac{r^2}{r^2+1}$ and $z = 1$. We discard $z = 1$ as being an obvious solution (see the above figure) and keep $z = \dfrac{r^2}{r^2+1} = \dfrac{x_1^2 + y_1^2}{x_1^2 + y_1^2 + 1}$ (this is called z' in the problem statement).

Now referring to $x = x_1[1 - z]$ and $y = y_1[1 - z]$ (at the top of this page] and using $z = z'$ (just found) we have: $x' = x_1[1 - z'] = x_1\left[1 - \dfrac{x_1^2 + y_1^2}{x_1^2 + y_1^2 + 1}\right]$

Similarly $y' = y_1[1 - z'] = y_1\left[1 - \dfrac{x_1^2 + y_1^2}{x_1^2 + y_1^2 + 1}\right]$.

b) $x_1^2 + y_1^2 = r^2$

Using the equations from (a):

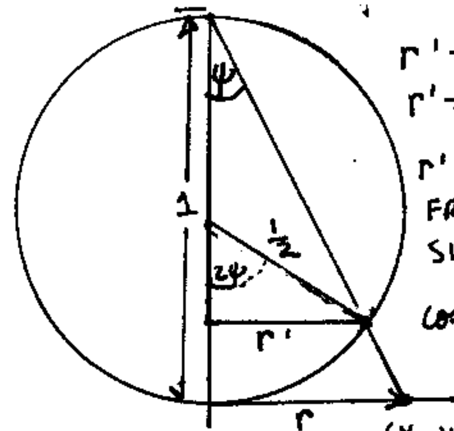
$$x' = x_1\left[1 - \frac{r^2}{1+r^2}\right] = \frac{x_1}{1+r^2}, \quad y' = \frac{y_1}{1+r^2}$$

$$x'^2 + y'^2 = \frac{x_1^2}{(1+r^2)^2} + \frac{y_1^2}{(1+r^2)^2} = \frac{x_1^2 + y_1^2}{(1+r^2)^2} = \left(\frac{r}{1+r^2}\right)^2$$

The radius of the circle on the sphere is $\dfrac{r}{1+r^2}$

cont'd next pg.

36 (b). continued



$r' = ?$

$r' = \frac{1}{2}\sin(2\psi)$

$r' = \sin\psi\cos\psi$

FROM this FIG.

$\sin\psi = \frac{r}{\sqrt{1+r^2}}$

$\cos\psi = \frac{1}{\sqrt{1+r^2}}$

(x_1, y_1) in complex z plane

$r' = \sin\psi\cos\psi = \frac{r}{1+r^2}$ q.e.d.